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PROJECTIVE REGULAR RESOLUTIONS OF SEMIMODULES

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Abstract: In this paper, we introduce projective regular resolutions of semimodules and prove the comparison theorem for those resolutions. Consequently, we construct cohomology monoids of a semimodule having a projective regular resolution, as well as calculate cohomology monoids of some special semimodules.

Keywords: Projective regular resolutions, cohomology monoid, semimodules

1 Introduction

Semirings, introduced by Vandiver [9] in 1934, generalize the notion of noncommutative rings in the sense that there do not have to exist negative elements. And we have the same sense for semimodules. Since then, there has been an active area of research in semirings and semimodules, both on the theoretical side and on the side of the application. The reader may consult the monographs of [2], e.g., for a more elaborate introduction to semirings and semimodules. Nowadays, the theory of semimodules is a subject study from many various aspects due tothe two following reasons.First, additively idempotent semirings are main subjects of researchin tropical geometry and linear algebra theory over tropical algebra. Second, in 1956, Tits introduced and studied field theory of one-element field, appearing from proving Riemann hypothesis for zeta-functions of curves over finite fields by Andre Weil, to intending to apply Weil's idea about proving Riemann hypothesis for the ring of integers. In this theory, an extension of one-element field is just an additively idempotent semiring.

As we haveknown, the homological theory is an important research tool not only in algebra but also in others, such asmodules where one defines homological and cohomological invariants based on projective resolutions. That is why, in a natural manner, we may consider semirings through semirings homological theory. Since the presentable category of a semiring (so called the semimodule category over a semiring) is a non-abelian one, the homological theory of semimodules over semirings is constructed inthe following different ways. Once, Janelidze [4] constructed cohomology monoids for generally non-additive categories with thev iewpoint of the extension theory, and then by applying these monoids for the category of semimodules, we obtain semimodules category cohomology monoids. But, we have great difficulties to calculate these cohomology monoids. Besides, Patchkoria [5], Tuyen [8], and Il'in [3] consider cohomology monoids of semimodules as thoseof complexes. We can refer [7] and

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references in [7] for reader's good understanding. However, up-to-date by above manner, we may only define cohomology monoids for some special classes of semimodules. Notice that for abelian categories having enough projective objects or injective ones (for example, the category of modules over rings), two above manners give us the equal cohomology monoids (in this case, cohomology groups), but unlikely for the category of semimodules, it is not the case. In 2003, Patchkoria [6] calculated the first cohomology monoids of semimodules by the first manner, based on the idea of projective resolutions. With the same idea, in this paper, we introduce the concept of a projective regular resolution of a semimoduleand prove the comparison theorem for these resolutions (Theorem 2.6). As a consequence, for any semimodule having a projective regular resolution, we construct the cohomology monoids with coefficients in an arbitrary semimodule (Theorem 2.7). Besides, we establish some their essential properties and calculate some specific examples of cohomology monoids based on projective regular resolutions.

The paper is organized as follows: Besides Introduction, in Section 1, for the reader's convenience, we include all subsequently necessary notions and facts on semirings and semimodules; in Section 2, we consider projective regular resolutions and the comparison theorem for these resolutions and we also construct cohomology monoids of semimodules with coefficients in semimodules, owing to projective regular resolutions.Finally, for notions and facts from semirings and semimodules, we refer to [2].

2 Preliminaries

Recall from [2] that a *semiring* Λ is an algebra (Λ, +, *0*) such that the following conditions are satisfied:

(1) (Λ, +, *0*) is a commutative monoid with identity element 0;

(2) (Λ, ., 1_Λ) is a monoid with identity element 1_Λ;

(3) Multiplication distributes over addition on either side;

(4) $0r = r0 = 0$ for all $r \in \Lambda$;

(5)$1_\Lambda \neq 0$.

As usual, a *left* Λ*-semimodule* over a semiring Λ is a commutative monoid $(M, +, 0_M)$ together with a scalar multiplication $(r,m) \mapsto rm$ from$\Lambda \times M \to M$ to M that satisfies the following identities for all $r,r' \in \Lambda$and$m,m' \in M$:

(1) $(rr')m = r(r'm)$;

(2) $r(m+m') = rm + rm'$;

(3) $(r+r')m = rm + r'm$;

(4) $r0_M = 0_M = 0m$.

Right Λ*-semimodules* and Λ*-homomorphisms* between semimodules are defined in the standard manner. An *image* of a Λ-homomorphism *f: A→ B* is a subsemimodule *Im(f)*

$=\{b \in B \mid b+f(a)= f(a')$ for some $a, a' \in A\}$. The *Kernel* of a Λ-homomorphism of semimodules f is defined as modules. Let $f: A \to B$ be a homomorphism of semimodules, f is called *k-regular* if $f(a_1) = f(a_2)$ then $a_1+k_1 = a_2+k_2$ for some k_1, k_2 in $Ker(f)$; f is called *i-regular* if $f(A) = Im(f)$; f is called *regular* if f is k-regular and i-regular.

A sequence of semimodules and homomorphisms $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is called ***exact*** if $Im(\alpha) = Ker(\beta)$.

Definition 1.1 [1]. A left Λ-semimodule P is ***projective*** if and only if the following condition holds: if $\varphi: M \to N$ is a surjective Λ-homomorphism of left Λ-semimodules and if $\alpha: P \to N$ is a Λ-homomorphism, then there exists a Λ-homomorphism $\beta: P \to M$ satisfying $\varphi\beta = \alpha$.

From the above condition in Def. 1.1 it follows that if $\varphi: M \to N$ is a k-regular Λ-homomorphism and if $\alpha, \alpha': P \to M$ are Λ-homomorphisms satisfying $\varphi\alpha = \varphi\alpha'$, then there are Λ-homomorphisms $\beta,\beta': P \to M$ such that $\alpha + \beta = \alpha' + \beta'$ and $\varphi\beta = \varphi\beta' = 0$.

Theorem 1.2 [1]. *A left Λ-semimodule P is projective if and only if P is a retract of a free Λ-semimodule.*

Definition 1.3 [8].

(i) For any semiring Λ, a ***chain complex*** C of Λ-semimodules is a family $\{C_n, \partial_n\}$ of Λ-semimodules C_n and Λ-semimodule homomorphisms $\partial_n: C_n \to C_{n-1}$, defined for all integers n, $-\infty < n < \infty$, and such that $\partial_n \partial_{n+1} = 0$. This last condition is equivalent to the statement that $\mathrm{Im}(\partial_{n+1}) \subset Ker(\partial_n)$ for all $n \in \mathbf{Z}$. If, for every $n \in \mathbf{Z}$, we set

$$H_nS(C) = Ker(\partial_n)/Im\,\partial_{n+1}) \cong Ker(\partial_n)/\partial_{n+1}(C_{n+1}),$$

then $H_nS(C)$ is called an n-th ***homology semimodule*** of a complex (C, ∂). The complex C is ***projective*** if each C_n is projective.

(ii) If (C, ∂) and (C', ∂') are complexes, a ***chain transformation*** $f: C \to C'$ is a family of Λ-semimodule homomorphisms $f_n: C_n \to C'_n$, one for each n, such that $\partial'_n f_n = f_{n-1}\partial_n$ for all n.

(iii) A ***chain homotopy***, written as $s: f \approx g$, between two chain transformations $f, g: C \to C'$ is a family of Λ-semimodule homomorphisms $s_n: C_n \to C'_{n+1}$ for each dimension n, such that.

$$\partial'_{n+1} s_n + s_{n-1}\partial_n + f_n = g_n. \qquad (*)$$

(iv) A sequence $\{d_n\}$ of semimodule homomorphisms is called ***regular*** if each d_n is regular.

(v) A ***complex*** (X, ε) ***over*** a ***left*** Λ***-semimodule*** M is a sequence of left Λ-semimodules X and Λ-homomorphisms.

$$\ldots \to X_n \xrightarrow{\partial_n} X_{n-1} \to \ldots \to X_1 \xrightarrow{\partial_1} X_0 \xrightarrow{\varepsilon} M \to 0, \qquad (**)$$

such that the composite of any two successive homomorphisms is zero.

3 On projective regular resolutions

Definition 2.1. A ***chain homotopy pair***, written as $(s,t){:}f \approx g$, between two chain transformations f, $g{:}\ C \to C'$is two families of Λ-semimodule homomorphisms s_n, $t_n{:}\ C_n \to C'_{n+1}$ for each dimension n, such that

$$\partial'_{n+1} s_n + s_{n-1} \partial_n + f_n = \partial'_{n+1} t_n + t_{n-1} \partial_n + g_n. \quad (*)$$

Remark 2.2. From Def. 2.1, when $t = 0$, we obtain Def.1.3.(iii).

Remark 2.3.

(i)We see that if f: $C \to C'$ is a chain transformation, then

$$H_nS(f){:}H_nS(C) \to H_nS(C')$$

$$[c_n] \mapsto [f_n(c_n)]$$

is a Λ-semimodule homomorphism. Also, if $f \approx g$ then $H_nS(f) = H_nS(g)$ for all $n \in \mathbf{Z}$.

(ii) Besides, let f: $C \to C'$ and f': $C' \to C$ be chain transformations such that $f \circ f' \approx 1_{C'}$ and $f' \circ f \approx 1_C$. We have $H_nS(C) \cong H_nS(C')$ for all integers$n$. Let (X, ∂) be a complex of left Λ-semimodules and M be a left Λ-semimodule. We then have the following cochain complex $Hom_\Lambda S(X, M)$ of commutative monoids:

$$\ldots \to Hom_\Lambda S(X_{n-1}, M) \xrightarrow{\delta^{n-1}} Hom_\Lambda S(X_n, M) \xrightarrow{\delta^n} Hom_\Lambda S(X_{n+1}, M) \to \ldots,$$

where $\delta^n(f) = f\partial_{n-1}$ for all $n \in \mathbf{Z}$.For any $n \in \mathbf{Z}$, the following commutative monoid

$$H^nS(X,M) := Ker(\delta^n)/Im(\delta^{n-1}) \cong Ker(\delta^n)/\delta^{n-1}(Hom_\Lambda S(X_{n-1}, M))$$

is called an n-***cohomology monoid of*** *the* ***complex*** X ***with coefficients in*** *the* ***left*** Λ***-semimodule*** M**.**

Remark 2.4.

(i) Coming back to modules, it is easy to prove that both an n-cohomology group of X with coefficients in M, as described above, and a well-known one for modules give the same result.

(ii) Let (X, ∂), (X', ∂') be complexes of left Λ-semimodules and, let f, $g{:}\ X \to X'$ be chain transformations such that $f \approx g$. Then

$$f^* \approx g^*{:}Hom_\Lambda S(X', M) \to Hom_\Lambda S(X, M).$$

Indeed, since f is chain-homotopic to g, there exists a family of homomorphic pairs $s_n, t_n{:}\ X_n \to X'_{n-1}$ such that every pair, for each dimension n, satisfies the equality (*). Next, we can show that

$$\delta^{n-1} s^*_{n-1} + s^*_n (\delta')^n + f^*_n = \delta^{n-1} t^*_{n-1} + t^*_n (\delta')^n + g^*_n.$$

For all $h \in Hom_\Lambda S(X_n, M)$ and for all $n \in \mathbf{Z}$, we can write

$$
\begin{aligned}
(\delta^{n-1}s^*_{n-1} + s^*_n(\delta')^n + f^*_n)(h) &= \delta^{n-1}s^*_{n-1}(h) + s^*_n(\delta')^n(h) + f^*_n(h) \\
&= \delta^{n-1}(s^*_{n-1}(h) + s^*_n(\delta')^n(h)) + f^*_n(h) \\
&= \delta^{n-1}(hs_{n-1}) + s^*_n(h\delta'_{n+1}) + hf_n \\
&= hs_{n-1}(\partial_n) + h\partial'_{n+1}(s_n) + hf_n \\
&= h(s_{n-1}\partial_n + \partial'_{n+1}s_n + f_n) \\
&= h(t_{n-1}\partial_n + \partial'_{n+1}t_n + f_n) \\
&= (\delta^{n-1}t^*_{n-1} + t^*_n(\delta')^n + g^*_n)(h).
\end{aligned}
$$

Therefore, $f^* \approx g^*$.

Specifically, we give several important concepts defined as follows.

Definition 2.5.

(i) A ***complex*** (X,ε) ***over*** a ***left*** Λ-***semimodule***M is a sequence of left Λ-semimodules X_n and Λ-homomorphisms

$$\ldots \to X_n \xrightarrow{\partial_n} X_{n-1} \to \ldots \to X_1 \xrightarrow{\partial_1} X_0 \xrightarrow{\varepsilon} M \to 0, \qquad (**)$$

such that the composite of any two successive homomorphisms is zero;

(ii) If the regular sequence (**) is exact, then it is called a ***regular resolution*** of M.

As well as modules, we also wish to compare a projective complex with a resolution; this is given in the following theorem.

Theorem 2.6 (Comparison Theorem). *Letγ: $M \to M'$ be a homomorphism of left Λ-semimodules, ε: $X \to M$ be a projective complex over M, and ε': $X' \to M'$ be a regular resolution of M'. Then, there is a transformation f: $X \to X'$ with$\varepsilon' f = \gamma\varepsilon$and any such two chain transformations are chain-homotopic.*

Proof: With the regularity of the homomorphisms δ_n, where $n = 0,1,2...$, ε', and by a proof similar to that for modules, we can directly obtain a chain transformation f: $X \to X'$ that satisfies $\varepsilon' f = \gamma\varepsilon$.

Next, suppose that there are two chain transformations f and g as stated in Theorem 2.6. We will prove that they are chain-homotopic, i.e. $f \approx g$. By choosing $t_{-2} = s_{-2} = t_{-1} = s_{-1} = 0$, we have

$\varepsilon' s_{-1} + s_{-2}\delta_{-1}+\gamma = \varepsilon' t_{-1} + t_{-2}\delta_{-1}+\gamma$. Suppose by induction that we have: s_0, t_0: $X_0 \to X'_1$, s_1, t_1: $X_1 \to X'_2$ ... and s_n, t_n: $X_n \to X'_n$, as in Theorem 2.6, such that

$$\partial'_{k+1} s_k + s_{k-1}\partial_k + f_k = \partial'_{k+1} t_k + t_{k-1}\partial_k + g_k \tag{1}$$

where $k = 0,1,..., n$. Since f, g are two chain transformations, we observe that

$$\partial'_{n+1} f_{n+1} = f_n \partial_{n+1} \text{ and } \partial'_{n+1} g_{n+1} = g_n \partial_{n+1},\ \forall n \in \mathbb{N}. \tag{2}$$

According to (1), we have

$$\partial'_{n+1} s_n + s_{n-1}\partial_n + f_n = \partial'_{n+1} t_n + t_{n-1}\partial_n + g_n$$

$$\Rightarrow \partial'_{n+1} s_n(\partial_{n+1}) + s_{n-1}\partial_n(\partial_{n+1}) + f_n \partial_{n+1} = \partial'_{n+1} t_n(\partial_{n+1}) + t_{n-1}\partial_n(\partial_{n+1}) + g_n \partial_{n+1}$$

$$\Rightarrow \partial'_{n+1}(s_n\partial_{n+1}) + \partial'_{n+1} f_{n+1} = \partial'_{n+1}(t_n\partial_{n+1}) + \partial'_{n+1} g_n$$

$$\Rightarrow \partial'_{n+1}(s_n\partial_{n+1} + f_{n+1}) = \partial'_{n+1}(t_n\partial_{n+1} + g_{n+1}). \tag{3}$$

Since ∂'_{n+1} is a k-regular Λ-homomorphism, $s_n\partial_{n+1} + f_{n+1}, t_n\partial_{n+1} + g_{n+1}: X_{n+1} \to X'_{n+1}$ is Λ-homomorphisms satisfying (3), and X_{n-1} is projective, there are Λ-homomorphisms k_{n+1}, h'_{n+1}: $X_{n+1} \to X'_{n+1}$ such that

$$\begin{gathered} s_n\partial_{n+1} + f_{n+1} + k_{n+1} = t_n\partial_{n+1} + g_{n+1} + k'_{n+1}{}' \\ \partial'_{n+1}k_{n+1} = \partial'_{n+1}k'_{n+1} = 0. \end{gathered} \tag{4}$$

This yields

$$k_{n+1}(X_{n+1}) \subset Ker(\partial'_{n+1}) = \partial'_{n+2}(X'_{n+2}),$$

$$k'_{n+1}(X_{n+1}) \subset Ker(\partial'_{n+1}) = \partial'_{n+2}(X'_{n+2}).$$

Then, k_{n+1+}, k'_{n+1}: $X_{n+1} \to \partial'_{n+2}(X'_{n+2})$. Since ∂'_{n+2}: $X'_{n+2} \to \partial'_{n+2}(X'_{n+2})$ is a surjective Λ-homomorphism and X_{n+1} is projective, there are Λ-homomorphisms s_{n+1}, t_{n+1}: $X_{n+1} \to X'_{n+2}$ such that

$$k_{n+1} = \partial'_{n+2}s_{n+1} \text{ and } k'_{n+1} = \partial'_{n+2}s_{n+1}. \tag{5}$$

Replace k_{n+1} and k'_{n+1} in (4) with the values given in (5) to obtain

$$\partial'_{n+2}s_{n+1} + s_n\partial_{n+1} + f_{n+1} = \partial'_{n+2}t_{n+1} + t_n\partial_{n+1} + g_{n+1}.$$

By induction, we then have $f \approx g$.

Using Theorem 2.6, and Remarks 2.3.(ii) and 2.4, we obtain the following important results.

Theorem 2.7. *If X and X′are two projective regular resolutions of a left Λ-semimodule M, and A is any left Λ-semimodule, then H^nS (X,A) $H^nS(X',A)$ depends only on M and A.*

Proof: Let (X, ∂), (X', ∂') be two projective regular resolutions of a left Λ-semimodule M and A be any left Λ-semimodule. From Remark 2.4.(ii), for two chain transformations such that $f \approx g$ we have the following chain homotopy pair:

$$f^* \approx g^*: Hom_\Lambda S\,(X', A) \to Hom_\Lambda S\,(X, A).$$

According to Theorem 2.6, if $\gamma = 1_M: M \to M$ is a homomorphism of left Λ-semimodules, $\varepsilon: X \to M$and $\varepsilon': X' \to M$are projective regular resolutions of M,then there is a transformation $f: X \to X'$ with $\varepsilon' f = \gamma\varepsilon$ and that any such two chain transformations are chain-homotopic. It follows that

$$H^n{}_\Lambda S(X',A) = H^n{}_\Lambda S(X,A).$$

It is observed from Theorem 2.7 that an n-cohomology monoid of a projective regular resolution X of a left Λ-semimodule M with coefficients in a left Λ-semimodule A does not depend on the choice of X, but only depends on M and A. Therefore, we may write

$$H^nS(M,A)= H^nS(X,A), \text{ where } n = 0,1,2,...$$

and we call an ***n-cohomology monoid*** of a ***left Λ-semimodule*** *M* with ***coefficients* in** a ***left Λ-semimodule*** *A***.**

As an example, we give the following statement that can be proved in a similar manner as for modules.

Proposition 2.8 - Example. *If Mis a projective left Λ-semimodule, and A is any left Λ-semimodule, then$H^nS(M,A) = 0$ for all$n > 0$, and$H^0S(M,A) \cong Hom_\Lambda S(M,A)$.*

Next, we give some other examples.

Example 2.9. Compute $H^n{}_{\Lambda_3} S(\Lambda_3/\{0,a\},\Lambda_3)$. Here, $\Lambda_3 = \{0,1,a)$ is a semiring with the addition and the multiplication being defined as follows:

+	0	1	a
0	0	1	a
1	1	1	1
a	a	a	a

×	0	1	a
0	0	0	0
1	0	1	a
a	0	a	a

Consider the following projective regular resolution of the Λ_3-semimodule $\Lambda_3/\{0,a\}$:

$$\ldots \to 0 \to \{0,a\} \xrightarrow{j} \Lambda_3 \xrightarrow{p} \Lambda_3/\{0,a\} \to 0,$$

where j is the normal injection and p is the normal projection. By applying Comparison Theorem, we obtain $H^n{}_{\Lambda_3} S(\Lambda_3/\{0,a\},\Lambda_3) = 0$ for all $n \geq 1$ and $H^0{}_{\Lambda_3} S(\Lambda_3/\{0,a\},\Lambda_3) \cong Hom_{\Lambda_3} S(\Lambda_3/\{0,a\},\Lambda_3)$.

Example 2.10. Compute $H^nS(\mathbb{N}/m\mathbb{N}, \mathbb{N})$ for all $m \in \mathbb{N}$. Consider the following projective regular resolution of ℕ-semimodule ℕ/mℕ:

$$\ldots \to 0 \to \mathbb{N} \xrightarrow{\alpha} \mathbb{N} \xrightarrow{p} \mathbb{N}/m\mathbb{N} \to 0,$$

where $\alpha(n) = nm$ for all $n \in \mathbb{N}$, and p is the normal projection. By applying the Comparison Theorem, we obtain

$$H^0S(\mathbb{N}/m\mathbb{N}, \mathbb{N}) \cong Hom_{\mathbb{N}}(\mathbb{N}/m\mathbb{N}, \mathbb{N}),$$

$$H^1S(\mathbb{N}/m\mathbb{N}, \mathbb{N}) = Hom_{\mathbb{N}}(\mathbb{N}, \mathbb{N}/\alpha^*(Hom_{\mathbb{N}}(\mathbb{N},\mathbb{N}))) \cong \mathbb{N}/m\mathbb{N},$$

$$H^nS(\mathbb{N}/m\mathbb{N}, \mathbb{N}) = 0, \forall n \geq 2.$$

Next, we consider cohomology monoids of semimodules in relation with their direct sum and product.

Proposition 2.11. *Let $\varepsilon: X \to M$ be a projective regular resolution of a left Λ-semimodule M, and $(A_i)_{i\in I}$ a family of left Λ-semimodules. Then,*

$$H^nS(M,\prod_{i\in I} A_i) \cong \prod_{i\in I} H^nS(M,A_i).$$

Proof: For any $\overline{\alpha} \in H^nS(M,\prod_{i\in I} A_i)$, we have $\delta^{n+1}_{\prod_{i\in I} A_i}(\alpha)=0$. It implies that $\alpha\delta_{n+1}=0$. Then, for all $i \in I$, we have

$$p_{Ai}(\alpha\delta_{n+1}) = 0,$$

$$\Rightarrow (p_{Ai}\alpha)\partial_{n+1} = 0,$$

$$\Rightarrow \delta^{n+1}_{A_i}(P_{A_i}\alpha)\partial_{n+1} = 0,$$

$$\Rightarrow P_{A_i}\alpha \in H^nS(M,A_i).$$

This gives the following

$$f: H^nS(M,\prod_{i\in I}A_i) \to \prod_{i\in I}H^nS(M, A_i)$$

$$\overline{\alpha} \qquad \mapsto (\overline{p_{A_i}\alpha})_{i\in I}.$$

If $\overline{\alpha} = \overline{\beta} \in H^nS(M,\prod_{i\in I}A_i)$, then for some $a,b \in Hom_\Lambda(X_{n-1}, \prod_{i\in I}A_i)$,

$$\alpha + \delta^{n+1}_{\prod_{i\in I} A_i}(a) = \beta + \delta^{n+1}_{\prod_{i\in I} A_i}(b),$$

which gives

$$\alpha + a\partial_n^{\prod A_i} = \beta + b\partial_n^{\prod A_i}.$$

Then, for all $i \in I$, we have

$$p_{A_i}(\alpha) + p_{A_i}(a\partial_n^{\prod A_i}) = p_{A_i}(\beta) + p_{A_i}(b\partial_n^{\prod A_i})$$

$$\Rightarrow p_{A_i}\alpha + \delta^n_{\prod_{i\in I} A_i}(p_{A_i}, a) = p_{A_i}\beta + \delta^n_{\prod_{i\in I} A_i}(p_{A_i} b),$$

$$\Rightarrow \overline{p_{A_i}\alpha} = \overline{p_{A_i}\beta}.$$

Therefore, $f(\overline{\alpha}) = f(\overline{\beta})$. Hence, f is a mapping. It can be checked directly that f is a homomorphism of monoids.

Now we show that f is a surjective homomorphism. Indeed, we have $\delta^{n+1}_{\prod_{i\in I} A_i}(\alpha)=0$ for all $(\overline{\alpha_i})_{i\in I} \in \prod_{i\in I} H^n S(M,A_i)$. Consider the following Λ-homomorphism:

$$\alpha : X_n \to \prod_{i\in I} A_i$$

$$x \mapsto (\alpha_i(x_i))_{i\in I.}$$

Then,

$$\delta^{n+1}_{\prod_{i\in I} A_i}(\alpha) = \alpha\partial^{\prod A_i}_{n+1} = (\alpha_i \partial^{A_i}_{n+1})_{i\in I} = 0,$$

which results in $\overline{\alpha} \in H^n S(M, \prod_{i\in I} A_i)$ and $f(\overline{\alpha}) = \overline{\alpha}_{i\in I}$. Thus, f is surjective. It can be shown, by an analogous manner, that f is also injective. Hence, f is isomorphic.

Proposition 2.12. *Let A be a left Λ-semimodule, $(M_i)_{i\in I}$ be a family of left Λ-semimodules, and X^i be a projective regular resolution of M_i. Then,*

$$H^n S(\bigoplus_{i\in I} M_i, A) \cong \prod_{i\in I} H^n S(M_i, A).$$

Proof: Let $\varepsilon_i : X^i \to M_i$ be a projective resolution of M_i. Then,

$$\bigoplus_{i\in I} \varepsilon_i : \bigoplus_{i\in I} X^i \to \bigoplus_{i\in I} M_i$$

is also a projective regular resolution of $\bigoplus_{i\in I} M_i$. For a Λ-homomorphism

$$\bigoplus_{i\in I} \alpha_i : \bigoplus_{i\in I} M_i \to A$$

$$(x_i)_I \mapsto \sum \alpha_i(x_i) \;\; ((x_i)_I \text{ has a finite support}),$$

we have the following group homomorphism

$$f : \prod_{i\in I} H^n S(M_i, A) \to H^n S(\bigoplus_{i\in I} M_i, A)$$

$$(\overline{\alpha_i})_I \mapsto \overline{\bigoplus_{i\in I} \alpha_i}.$$

For all $\overline{\alpha_i} \in H^n S(\oplus_{i\in I} M_i, A)$ such that $\alpha(\oplus_{i\in I} \partial^i_{n+1}) = 0$, we set $\alpha_i = \alpha i_{M_i} : M_i \to A$, where $i_{M_i}: M_i \to \oplus_{i\in I} M_i$ is an injection. Since all $(0, ..., o, x_i, 0, ...) \in \alpha(\oplus_{i\in I} M_i), \alpha(\oplus_{i\in I} \partial^i_{n+1}) = 0$, we obtain

$$(\alpha . \bigoplus_{i\in I} \partial^i_{n+1})((0,...,0,x_i,0,...)) = 0$$

$$\Rightarrow \alpha((0,...,0,\partial^i_{n+1}(x_i),0,...)) = 0 \Rightarrow \alpha(i_{M_i}(\partial^i_{n+1}(x_i))) = 0$$

$$\Rightarrow \alpha . i_{M_i} \partial^i_{n+1} = 0 \Rightarrow \alpha_i . \partial^i_{n+1} = 0 \Rightarrow \overline{\alpha_i} \in H^n S(M_i, A).$$

Then, $f((\overline{\alpha_i})_I = \overline{\alpha}$, i.e f is surjective.

Now, we verify injectivity of f. For all $(\overline{\alpha_i})_I$, $(\overline{\beta_i})_I \in \prod_{i\in I} H^n S(M_i, A)$ satisfying $f((\overline{\alpha_i})_I = f((\overline{\beta_i})_I)$, the following equality holds: $\overline{\oplus_{i\in I}\alpha_i} = \overline{\oplus_{i\in I}\beta_i}$. It implies that

$$\bigoplus_{i\in I} a_i + (\bigoplus_{i\in I}\partial_n^i)^*(a) = \bigoplus_{i\in I}\beta_i + (\bigoplus_{i\in I}\partial_n^i)^*(b)$$

for some $a,b \in HomS(\oplus_{i\in I} X_{n-1}^i, A)$. Then,

$$\bigoplus_{i\in I} a_i + a(\bigoplus_{i\in I}\partial_n^i) = \bigoplus_{i\in I}\beta_i + b(\bigoplus_{i\in I}\partial_n^i).$$

Moreover, for all $x_i \in M_i$, we have $x = (0,...,0.x_i,0,...) \in \oplus_{i\in I} M_i$. Therefore,

$$(\bigoplus_{i\in I}\alpha_i + a(\bigoplus_{i\in I}\partial_n^i))(x) = (\bigoplus_{i\in I}\beta_i + b(\bigoplus_{i\in I}\partial_n^i))(x)$$

$$\Rightarrow \alpha_i(x_i) + ai_M(\partial^i_n(x_i)) = \beta_i(x_i) + bi_M(\partial^i_n(x_i))$$

$$\Rightarrow \alpha_i + a.i_{M_i}(\partial_n^i) = \beta_i + bi_{M_i}\partial_n^i \Rightarrow \overline{\alpha_i} = \overline{\beta_i}\ \forall i\in I \Rightarrow (\overline{\alpha_i})_I = (\overline{\beta_i})_I.$$

Hence, f is injective.

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