Hue University Journal of Science: Natural Science; ISSN 1859 –1388

Vol. 127, No. 1A, 2018, P. 125 – 133; DOI: 10.26459/hueuni-jns.v127i1A.4462

$g*s$-IRRESOLUTE MAPS IN TOPOLOGICAL SPACES

Ong Van Tuyen*

Ong Ich Khiem high school, Hoa Phong, Hoa Vang, Da Nang, Vietnam

Abstract. In 2009, El-Maghrabi and Nasef introduced a new class of sets between semi-closed and gs-closed sets called $g*s$-closed and gave some of their properties. In this paper, we introduce and study the concepts of three new classes of maps, namely $g*s$-continuous maps, $g*s$-irresolute maps and $g*s$-closed maps. Moreover, we introduce the concepts of $g*s$-compactness of the topological spaces.

Keywords: $g*s$-closed sets, $g*s$-compact sets, $g*s$-continuous maps, $g*s$-irresolute maps, $g*s$-closed maps.

1. Introduction

In 1963, Levine introduced and studied the concepts of semi-open sets and semi-continuous maps [2]. Later, Crossley and Hildebrand gave the concept of irresolute maps [3]. In 1995, sg-irresolute maps and sg-continuous maps were introduced by Caldas [4]. Recently, El-Maghrabi and Nasef have introduced a new class of sets called $g*s$-closed and given some of their properties [1].

In this paper, we prove that $g*s$-closed and sg-closed (g-closed) sets are independent. Furthermore, we introduce and study the concepts of three new classes of maps, namely $g*s$-continuous maps, $g*s$-irresolute maps and $g*s$-closed maps. Finally, we introduce the concepts of $g*s$-compactness of the topological spaces. Throughout this paper, X, Y are topological spaces.

Denifition 1.1 [2]**.** A subset A of a space X is called

(1) *semi-open* if there exists an open set U such that $U \subset A \subset \overline{U}$.

(2) *semi-closed* if $X - A$ is semi-open.

Denifition 1.2 [2]**.** The intersection of all semi-closed sets containing A is called the *semi-closure* of A and is denoted by $s(\overline{A})$.

Denifition 1.3. A subset A of a space X is called

* *Corresponding: tuyenvan612dn@gmail.com*

Submitted: 3–8 –2017; Revised: 26–7– 2018; Accepted: 10–8– 2018

(1) g *-closed* [5] if $\overline{A} \subset G$ whenever $A \subset G$ and G is open in X.

(2) g *-open* [5] if $X - A$ is g -closed.

(3) gs *-closed* [6] if $s(\overline{A}) \subset G$ whenever $A \subset G$ and G is open in X.

(4) sg *-closed* [7] if $s(\overline{A}) \subset G$ whenever $A \subset G$ and G is semi-open in X.

(5) sg *-open* [7] if $X - A$ is sg -closed.

(6) $g*s$ *-closed* [1] if $s(\overline{A}) \subset G$ whenever $A \subset G$ and G is g -open in X.

(7) $g*s$ *-open* [1] if $X - A$ is $g*s$ -closed.

***Remark* 1.4.**

(1) Closed sets $\Rightarrow$ g -closed sets [5].

(2) Open sets $\Rightarrow$ g -open sets.

(3) Closed sets $\Rightarrow$ semi-closed sets $\Rightarrow$ $g*s$ -closed sets $\Rightarrow$ gs -closed sets [1].

(4) Open sets $\Rightarrow$ semi-open sets $\Rightarrow$ $g*s$ -open sets.

(5) Closed sets $\Rightarrow$ sg -closed sets [7].

(6) Open sets $\Rightarrow$ sg -open sets.

Denifition 1.5. A map $f : X \to Y$ is called

(1) *irresolute* [3] if $f^{-1}(U)$ is semi-open in X for every semi-open subset U of Y.

(2) sg *-irresolute* [4] if $f^{-1}(U)$ is sg -closed in X for every sg -closed subset U of Y.

(3) $g*s$ *-irresolute* if $f^{-1}(U)$ is $g*s$ -closed in X for every $g*s$ -closed subset U of Y.

Denifition 1.6. A map $f : X \to Y$ is called

(1) *semi-continuous* [2] if $f^{-1}(U)$ is semi-open in X for every open subset U of Y.

(2) sg *-continuous* [4] if $f^{-1}(U)$ is sg -closed in X for every closed subset U of Y.

(3) $g*s$ *-continuous* if $f^{-1}(U)$ is $g*s$ -closed in X for every closed subset U of Y.

Denifition 1.7. A map $f : X \to Y$ is called

(1) *semi-closed* [8] if for each closed subset F of X, $f(F)$ is semi-closed in Y.

(2) *semi-open* [8] if for each open subset F of X, $f(F)$ is semi-open in Y.

(3) $g*s$ *-closed* if for each closed subset F of X, $f(F)$ is $g*s$-closed in Y.

(4) $g*s$ *-open* if for each open subset F of X, $f(F)$ is $g*s$-open in Y.

Lemma 1.8 [6]. *A map $f : X \to Y$ is irresolute if and only if $f^{-1}(U)$ is semi-closed in X for every semi-closed subset U of Y.*

Lemma 1.9 [2]. *A map $f : X \to Y$ is semi-continuous if and only if $f^{-1}(U)$ is semi-closed in X for every closed subset U of Y.*

Lemma 1.10 [4]. *If a map $f : X \to Y$ is sg -irresolute, then it is sg -continuous but not conversely.*

Definition 1.11. Let $\mathcal{P}$ be a family of subsets of a space X and A be a subset of X. Then,

(1) $\mathcal{P}$ is called a $g*s$ *-open cover* of A if $\mathcal{P}$ is a cover of A and P is $g*s$-open whenever $P \in \mathcal{P}$.

(2) A is called $g*s$ *-compact* if every $g*s$-open cover of A has a finite subcover.

(3) X is called $g*s$ *-compact space* if it is a $g*s$-compact set.

2. Main results

***Remark* 2.1.**

(1) $g*s$-closed and sg-closed sets are independent.

(2) $g*s$-closed and g-closed sets are independent.

Proof. It can be seen in the following example.

***Exemple* 2.2.**

(1) Let $X = \{a,b,c\}$ with topology $\tau = \{\varnothing, \{a\}, \{a,c\}, X\}$. If $A = \{a,b\}$, then A is $g*s$-closed but not sg-closed.

(2) Let $X=\{a,b,c\}$ with topology $\tau=\{\varnothing,\{a\},\{b,c\},X\}$. If $A=\{b\}$, then A is sg-closed but not $g*s$-closed.

(3) Let $X=\{a,b,c\}$ with topology $\tau=\{\varnothing,\{a\},X\}$. If $A=\{a,b\}$, then A is g-closed but not $g*s$-closed.

(4) Let $X=\{a,b,c\}$ with topology $\tau=\{\varnothing,\{a\},\{a,c\},X\}$. If $A=\{c\}$, then A is $g*s$-closed but not g-closed.

Theorem 2.3. *A map $f:X\to Y$ is g*s-irresolute if and only if $f^{-1}(U)$ is $g*s$-open in X for every $g*s$-open subset U of Y.*

Proof. Necessity. If U is a $g*s$-open subset of Y, then $Y-U$ is $g*s$-closed in Y. Since f is $g*s$-irresolute, $f^{-1}(Y-U)$ is $g*s$-open in X. Moreover, since $X-f^{-1}(U)=f^{-1}(Y-U)$, hence $f^{-1}(U)$ is $g*s$-open in X.

Sufficiency. If B is a $g*s$-closed subset of Y, then $Y-B$ is $g*s$-open in Y. Moreover, since $f^{-1}(Y-B)=X-f^{-1}(B)$, $X-f^{-1}(B)$ is $g*s$-open in X. Hence $f^{-1}(B)$ is $g*s$-closed in X. Therefore, f is $g*s$-irresolute.

***Remark* 2.4.**

(1) $g*s$-irresolute and irresolute maps are independent.

(2) $g*s$-irresolute and sg-irresolute maps are independent.

Proof. It can be seen in the following example.

***Example* 2.5.**

(1) Let $X=\{a,b,c\}$ with topology $\tau=\{\varnothing,\{a\},X\}$ and $\sigma=\{\varnothing,\{a\},\{a,c\},X\}$. Then, the identity map $i:(X,\tau)\to(X,\sigma)$ is irresolute. However, $\{a,b\}$ is $g*s$-closed in (X,σ) by Example 2.2 (1) but is not $g*s$-closed in (X,τ). Therefore, f is not $g*s$-irresolute.

(2) Let $X=\{a,b,c,d\}$ with topology $\tau=\{\varnothing,\{c,d\},X\}$ and $Y=\{p,q\}$ with topology $\sigma=\{\varnothing,\{q\},Y\}$. Let the map $f:(X,\tau)\to(Y,\sigma)$ be defined by

$f(a) = f(b) = f(d) = p$ and $f(c) = q$. Then f is $g*s$-irresolute. However, $\{p\}$ is semi-closed in (Y,σ) but $f^{-1}(\{p\}) = \{a,b,d\}$ is not semi-closed in (X,τ). Therefore, f is not irresolute.

(3) Let $X = \{a,b,c\}$ with topology $\tau = \{\varnothing, \{a\}, \{b,c\}, X\}$. Let the map $f:(X,\tau) \to (X,\tau)$ be defined by $f(a) = f(c) = b$ and $f(b) = a$. Then, f is sg-irresolute. However, $\{a\}$ is $g*s$-closed in (X,τ) but $f^{-1}(\{a\}) = \{b\}$ is not $g*s$-closed in (X,τ). Therefore, f is not $g*s$-irresolute.

(4) Let $X = \{a,b,c\}$ with topology $\tau = \{\varnothing, \{a\}, \{a,c\}, X\}$ and $\sigma = \{\varnothing, \{a\}, \{b,c\}, X\}$. Let the map $f:(X,\tau) \to (X,\sigma)$ be defined by $f(a) = f(b) = c$ and $f(c) = b$. Then, f is $g*s$-irresolute. However, $\{c\}$ is sg-closed in (X,σ) but $f^{-1}(\{c\}) = \{a,b\}$ is not sg-closed in (X,τ). Therefore, f is not sg-irresolute.

Theorem 2.6. *A map $f: X \to Y$ is $g*s$-continuous if and only if $f^{-1}(U)$ is $g*s$-open in X for every open subset U of Y.*

Proof. Necessity. If U is an open subset of Y, then $Y - U$ is closed in Y. Since f is $g*s$-continuous, $f^{-1}(Y-U)$ is $g*s$-closed in X. Moreover, since $X - f^{-1}(U) = f^{-1}(Y-U)$, hence $f^{-1}(U)$ is $g*s$-open in X.

Sufficiency. If B is a closed subset of Y, then $Y - B$ is open in Y. Moreover, since $f^{-1}(Y-B) = X - f^{-1}(B)$, $X - f^{-1}(B)$ is $g*s$-open in X. Hence, $f^{-1}(B)$ is $g*s$-closed in X. Therefore, f is $g*s$-continuous.

Theorem 2.7. *If a map $f: X \to Y$ is $g*s$-irresolute or semi-continuous, then it is $g*s$-continuous but not conversely.*

Proof. (1) If U is a closed subset of Y, then U is $g*s$-closed in Y by Remark 1.4 (3). Moreover, since f is $g*s$-irresolute, $f^{-1}(U)$ is $g*s$-closed in X. Therefore, f is $g*s$-continuous.

(2) By Remark 1.4 (3), every semi-closed set is $g*s$-closed. It shows that if f is semi-continuous, then f is $g*s$-continuous.

(3) Let $X = \{a,b,c\}$ with topology $\tau = \{\varnothing, \{a\}, \{c\}, \{a,c\}, X\}$ and $\sigma = \{\varnothing, \{a\}, X\}$. Let

the map $f:(X,\tau)\to(X,\sigma)$ be defined by $f(a)=f(c)=b$ and $f(b)=c$. Then f is $g*s$-continuous. However, $\{b\}$ is $g*s$-closed in (X,σ) but $f^{-1}(\{b\})=\{a,c\}$ is not $g*s$-closed in (X,τ). Therefore, f is not $g*s$-irresolute.

(4) Let X,Y and f be as in Example 2.5 (2). Then, f is $g*s$-irresolute. Hence, f is $g*s$-continuous by (1). However, $\{p\}$ is closed in (Y,σ) but $f^{-1}(\{p\})$ is not semi-closed in (X,τ). Therefore, f is not semi-continuous.

***Remark* 2.8.** $g*s$-continuous maps and sg-continuous maps are independent.

Proof. It can be seen in the following example.

***Example* 2.9.**

(1) Let X and f be as in Example 2.5 (3). Then, f is sg-irresolute. By Lemma 1.12, f is sg-continuous. However, $\{a\}$ is closed in (X,τ) but $f^{-1}(\{a\})=\{b\}$ is not $g*s$-closed in (X,τ) by Example 2.2 (2). Therefore, f is not $g*s$-continuous.

(2) Let $X=\{a,b,c\}$ with topology $\tau=\{\varnothing,\{a\},\{a,c\},X\}$ and $Y=\{p,q\}$ with topology $\sigma=\{\varnothing,\{q\},Y\}$. Let the map $f:(X,\tau)\to(Y,\sigma)$ be defined by $f(a)=f(b)=p$ and $f(c)=q$. Then, f is $g*s$-continuous. However, $\{p\}$ is closed in (Y,σ) but $f^{-1}(\{p\})=\{a,b\}$ is not sg-closed in (X,τ) by Example 2.2 (1). Therefore, f is not sg-continuous.

Theorem 2.10. *If a map $f:X\to Y$ is semi-closed, then it is $g*s$-closed but not conversely.*

Proof. By Remark 1.4 (3), every semi-closed set is $g*s$-closed. It shows that f is $g*s$-closed.

The converse does not need to be true. In fact, let $X=\{a,b,c\}$ with topology $\tau=\{\varnothing,\{c\},X\}$ and $\sigma=\{\varnothing,\{a\},\{a,c\},X\}$. Then, the identity map $i:(X,\tau)\to(X,\sigma)$ is $g*s$-closed. However, $\{a,b\}$ is closed in (X,τ) but is not semi-closed in (X,σ). Therefore, f is not semi-closed.

Corollary 2.11. *If a map $f:X\to Y$ is semi-open, then it is $g*s$-open but not conversely.*

Theorem 2.12. *Let $f:X\to Y$ be a map. Then, the following are equipvalent*

(1) *f is $g*s$-closed;*

(2) *For each subset S of Y and for each open subset U in X such that $f^{-1}(S) \subset U$, there is a $g*s$-open subset V of Y such that $S \subset V$ and $f^{-1}(V) \subset U$;*

(3) *For each $y \in Y$ and for each open subset U in X such that $f^{-1}(y) \subset U$, there is a $g*s$-open subset V of Y such that $y \in V$ and $f^{-1}(V) \subset U$.*

Proof. (1) $\Rightarrow$ (2). Let f be $g*s$-closed, $S \subset Y$ and U be an open subset of X such that $f^{-1}(S) \subset U$. Since $X - U$ is closed in X and f is $g*s$-closed, $f(X-U)$ is $g*s$-closed in Y. Thus, $V = Y - f(X-U)$ is $g*s$-open in Y and

$$f^{-1}(V) = f^{-1}\left[Y - f(X-U)\right] = X - f^{-1}(f(X-U)) \subset X - (X-U) = U.$$

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1). Let A be a closed subset of X. For each $y \in Y - f(A)$, we have

$$f^{-1}(y) \subset f^{-1}\left[Y - f(A)\right] = X - f^{-1}(f(A)) \subset X - A$$

and $X - A$ is open in X. By hypothesis, there is a $g*s$-open subset V of Y such that $y \in V$ and $f^{-1}(V) \subset X - A$. Hence, $Y - f(A) \subset$ V and $A \subset X - f^{-1}(V) = f^{-1}(Y - V)$. Thus,

$$Y - V \subset f(A) \subset f(f^{-1}(Y-V)) \subset Y - V$$

which implies $Y - V = f(A)$. Since $Y - V$ is $g*s$-closed in Y, $f(A)$ is $g*s$-closed in Y. Therefore, f is a $g*s$-closed map.

Proposition 2.13. *Let $f: X \to Y$, $g: Y \to Z$ and $g \circ f: X \to Z$ be maps. Then,*

(1) *If f, g are both $g*s$-irresolute, then $g \circ f$ is $g*s$-irresolute.*

(2) *If f is $g*s$-irresolute and g is $g*s$-continuous, then $g \circ f$ is $g*s$-continuous.*

(3) *If f is closed and g is $g*s$-closed, then $g \circ f: X \to Z$ is $g*s$-closed.*

Proof. (1) If U is $g*s$-open in Z, then $g^{-1}(U)$ is $g*s$-open in Y and $f^{-1}(g^{-1}(U))$ is $g*s$-open in X because f, g are $g*s$-irresolute. Thus, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is $g*s$-open in X. Therefore, $g \circ f$ is $g*s$-irresolute.

(2) If U is closed in Z, then $g^{-1}(U)$ is $g*s$-closed in Y and $f^{-1}(g^{-1}(U))$ is $g*s$-closed

in X because f is $g*s$-irresolute and g is $g*s$-continuous. Thus, $(g\circ f)^{-1}(U)=f^{-1}(g^{-1}(U))$ is $g*s$-closed in X. Therefore, $g\circ f$ is $g*s$-continuous.

(3) Let U be closed in X. Since f is a closed map, $f(U)$ is closed in Y. Moreover, since g is a $g*s$-closed map, thus $(g\circ f)(U)=g(f(U))$ is $g*s$-closed in Z. Therefore, $g\circ f$ is $g*s$-closed.

Corollary 2.14. *If a map $f:X\to Y$ is open and a map $g:Y\to Z$ is $g*s$-open, then $g\circ f:X\to Z$ is a $g*s$-open map.*

Theorem 2.15. *Let $f:X\to Y$ be a map. Then,*

(1) *If f is $g*s$-continuous and X is a $g*s$-compact space, then Y is a compact space.*

(2) *If f is $g*s$-irresolute and B is $g*s$-compact in X, then $f(B)$ is $g*s$-compact in Y.*

Proof. (1) Let $\{A_s : s\in S\}$ be an open cover of Y. Then $\{f^{-1}(A_s): s\in S\}$ is a $g*s$-open cover of X because f is $g*s$-continuous. Since X is a $g*s$-compact space, it has a finite subcover, i.e., there exist $s_1,s_2,\dots,s_n\in S$ such that

$$X=\bigcup_{i=1}^{n} f^{-1}(A_{s_i})=f^{-1}\left(\bigcup_{i=1}^{n} A_{s_i}\right).$$

Thus,

$$Y=f\left(f^{-1}\left(\bigcup_{i=1}^{n} A_{s_i}\right)\right)=\bigcup_{i=1}^{n} A_{s_i}.$$

Therefore, Y is a compact space.

(2) Let $\{A_s : s\in S\}$ be a collection of $g*s$-open subsets of Y such that $f(B)\subset\bigcup_{s\in S} A_s$. Then,

$$B\subset f^{-1}(f(B))\subset f^{-1}\left(\bigcup_{s\in S} A_s\right)=\bigcup_{s\in S} f^{-1}(A_s).$$

By hypothesis, there exist $s_1,s_2,\dots,s_n\in S$ such that $B\subset\bigcup_{i=1}^{n} f^{-1}(A_{s_i})$. Thus,

$$f(B) \subset f\left(\bigcup_{i=1}^{n} f^{-1}(A_{s_i})\right) = \bigcup_{i=1}^{n} f(f^{-1}(A_{s_i})) \subset \bigcup_{i=1}^{n} A_{s_i}.$$

Therefore, $f(B)$ is $g*s$-compact in Y.

Theorem 2.16. *Every $g*s$-closed subset of a $g*s$-compact space is $g*s$-compact.*

Proof. Let A be a $g*s$-closed subset of a $g*s$-compact space X. Then, $X-A$ is $g*s$-open in X. Let $M=\{G_s : s\in S\}$ be a $g*s$-open cover of A. Put $M' = M\cup(X-A)$. Then M' is $g*s$-open cover of X, i.e., $X=\left(\bigcup_{s\in S} G_s\right)\cup(X-A)$. Moreover, since X is a $g*s$-compact space, there exists a finite subcover of $\{G_s : s\in S, X-A\}$, i.e., $X=G_{s_1}\cup G_{s_2}\cup\ldots\cup G_{s_m}\cup(X-A)$ with $G_{s_i}\in M$. Thus, $A\subset G_{s_1}\cup G_{s_2}\cup\ldots\cup G_{s_m}$ with $G_{s_i}\in M$. Therefore, A is $g*s$-compact.

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