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A COMPACT IMBEDDING OF RIEMANNIAN SYMMETRIC SPACES

Tran Dao Dong

HU – University of Education, 32 Le Loi St., Hue, Vietnam

Abstract. Let G be a connected real semisimple Lie group with finite center and θ be a Cartan involution of G. Suppose that K is the maximal compact subgroup of G corresponding to the Cartan involution θ. The coset space $\mathbf{X} = G/K$ is then a Riemannian symmetric space. In this paper, by choosing reduced root system $\Sigma' = \{\alpha \in \Sigma \mid 2\alpha \notin \Sigma; \alpha/2 \notin \Sigma\}$ instead of restricted root system Σ and using the action of the Weyl group, first we construct a compact real analytic manifold $\mathbf{X}'$ in which the Riemannian symmetric space G/K is realized as an open subset and that G acts analytically on it; then, we consider the real analytic structure of $\mathbf{X}'$ induced from the real analytic structure of A_{IR}, the compactification of the corresponding vectorial part.

Keywords: symmetric space, Weyl group, Cartan involution, compactification

1 Introduction

Let G be a connected real semisimple Lie group with finite center and $\mathfrak{g}$ be the Lie algebra of G. Denote the Cartan involution of G by θ and K the fixed points of θ. Then, K is a maximal compact subgroup of G, and the coset space $\mathbf{X} = G/K$ becomes a Riemannian symmetric space. We also denote by θ the Cartan involution of $\mathfrak{g}$, corresponding to the Cartan involution θ of G. It follows that $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$ into eigenspaces of θ, where $\mathfrak{k}$ is the Lie algebra of K.

Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{p}$ and $\mathfrak{a}^*$ be the dual space of $\mathfrak{a}$. The corresponding analytic subgroup A of $\mathfrak{a}$ in G is, then, called the vectorial part of X. For a non zero $\alpha \in \mathfrak{a}^*$, non zero eigenspace

$$\mathfrak{g}_\alpha = \{Y \in \mathfrak{g} \mid [H, Y] = \alpha(H)Y, \forall H \in \mathfrak{a}\}$$

is called the root space, and the corresponding α's the restricted root. Then, the set $\Sigma = \{\alpha \in \mathfrak{a}^* \mid \mathfrak{g}_\alpha \neq \{0\}, \alpha \neq 0\}$ defines a root system with the inner product induced by the Killing form $<,>$ of $\mathfrak{g}$. Moreover, Weyl group W of Σ is defined with normalizer $N_K(\mathfrak{a})$ of $\mathfrak{a}$ in K modulo the centralizer $M = Z_K(\mathfrak{a})$ of $\mathfrak{a}$ in K. It acts naturally on $\mathfrak{a}$ and coincides via this action with the reflection group of the root system Σ.

* *Corresponding:trdaodong@yahoo.com*

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Choose a fundamental system $\Delta = \{ \alpha_1,...,\alpha_l \}$ of Σ, where number l, which equals dim $\mathfrak{a}$, is called the split rank of the symmetric space X and denote the corresponding set of all restricted positive roots in Σ by Σ^+.

Denote the complexification of $\mathfrak{g}$ by $\mathfrak{g}_{\mathbf{C}}$ and $G_{\mathbf{C}}$ the corresponding analytic group. Let $\mathfrak{a}_{\mathbf{C}}$ be the complexification of $\mathfrak{a}$ and $A_{\mathbf{C}}$ be the analytic subgroup of $\mathfrak{a}_{\mathbf{C}}$ in $G_{\mathbf{C}}$. For each $a \in A_{\mathbf{C}}$ and $\alpha \in \Sigma$, we define $a^\alpha = e^{\alpha \log a} \in \mathbf{C}^* = \mathbf{C} \setminus \{0\}$ and consider

$$A_{\mathrm{IR}} = \{ a \in A_{\mathbf{C}} \mid a^\alpha \in \mathrm{IR}, \forall \alpha \in \Sigma \}.$$

Let $(\mathbf{C}^*)^\Sigma$ be the set of complexes $z = (z_\beta)_{\beta\in\Sigma}$, where $z_\beta \in \mathbf{C}^*$ and $\mathbf{C}\mathrm{IP}^1$ be the 1-dimensional complex projective space. Then, we can define

$$\varphi : A_{\mathbf{C}} \to (\mathbf{C}^*)^\Sigma, a \mapsto \varphi(a) = (a^\alpha)_{\alpha\in\Sigma}.$$

In [2], based on the natural embedding of $(\mathbf{C}^*)^\Sigma$ into $(\mathbf{C}\mathrm{IP}^1)^\Sigma$, we obtained an embedding of A_{IR} into a compact real analytic manifold $\mathcal{A}_{\mathrm{IR}}$ which is called a compactification of A_{IR}, and then constructed a realization of G/K in a compact real analytic manifold. In [3] , we applied the construction for semisimple symmetric spaces, and determined the system of invariant differential operators on the corresponding compactifications [4].

In this paper, by choosing reduced root system $\Sigma' = \{\alpha \in \Sigma \mid 2\alpha \notin \Sigma; \frac{\alpha}{2} \notin \Sigma\}$ instead of Σ and using the action of the Weyl group, first we construct a compact real analytic manifold $\mathbf{X}'$ in which the Riemannian symmetric space G/K is realized as an open subset and that G acts analytically on it; then, we consider the real analytic structure of $\mathbf{X}'$ induced from the real analytic structure of $\mathcal{A}_{\mathrm{IR}}$.

Our construction is a motivation for the construction of Oshima and Sekiguchi [7] for affine symmetric spaces and it is similar to those in [6], [8], [9] for semisimple symmetric spaces.

2 A compactification of the vectorial part

In this section, we recall some notations and results concerning compactification $\mathcal{A}_{\mathrm{IR}}$ of vectorial part A_{IR} constructed in [2] and then illustrate the construction via the case of symmetric space $SL(n,\mathrm{IR})/SO(n)$.

Let G be a connected real semisimple Lie group with a finite center and $\mathfrak{g}$ be the Lie algebra of G. Denote the complexification of $\mathfrak{g}$ by $\mathfrak{g}_{\mathbf{C}}$ and $G_{\mathbf{C}}$ the corresponding analytic group. For simplicity, we assume that G is a real form of complex Lie group $G_{\mathbf{C}}$. Let $\mathfrak{a}_{\mathbf{C}}$ be the complexification of $\mathfrak{a}$ and $A_{\mathbf{C}}$ be the analytic subgroup of $\mathfrak{a}_{\mathbf{C}}$ in $G_{\mathbf{C}}$. Then, we can consider the map $\varphi : A_{\mathbf{C}} \to (\mathbf{C}^*)^{\Sigma}$, which is defined with $\varphi(a) = (a^{\alpha})_{\alpha\in\Sigma}, \forall a \in A_{\mathbf{C}}$, where $(\mathbf{C}^*)^{\Sigma}$ is the set of complexes $z = (z_{\beta})_{\beta\in\Sigma}$.

It follows that for every $z = (z_{\alpha})_{\alpha\in\Sigma} \in \varphi(A_{\mathbf{C}})$, we have

$$z_{-\alpha} = (z_{\alpha})^{-1}, \forall \alpha \in \Sigma \tag{1}$$

$$z_{\alpha} = \prod_{\gamma\in\Delta}(z_{\gamma})^{k(\alpha,\gamma)}, \forall \alpha \in \Sigma^+, \alpha = \sum_{\gamma\in\Delta} k(\alpha,\gamma).\gamma. \tag{2}$$

Denote the 1-dimensional complex projective space by $\mathbf{C}\mathrm{IP}^1$. Then, based on the natural embedding of $(\mathbf{C}^*)^{\Sigma}$ into $(\mathbf{C}\mathrm{IP}^1)^{\Sigma}$, we get an embedding map of $A_{\mathbf{C}}$ into $(\mathbf{C}\mathrm{IP}^1)^{\Sigma}$ denoted also by φ.

Let $\mathbf{M} = \{z \in (\mathrm{IRIP}^1)^{\Sigma} \mid z_{-\alpha} = z_{\alpha}^{-1}, \forall \alpha \in \Sigma\}$. By definition, we see that $\mathbf{M}$ is compact. Moreover, subset

$$\mathcal{U}_{\Sigma^+} = \{ m = (m_{\alpha}, m_{-\alpha}) \in \mathbf{M} \mid m_{\alpha} \in \mathrm{IR}, m_{-\alpha} \in \mathrm{IR}^* \cup \{\infty\}, \forall \alpha \in \Sigma^+\}$$

is an open subset in $(\mathrm{IRIP}^1)^{\Sigma^+}$, and we get homeomorphism $\chi_{\Sigma^+} : \mathcal{U}_{\Sigma^+} \to \mathrm{IR}^{\Sigma^+}$ defined by $\chi_{\Sigma^+}(m) = (m_{\alpha})_{\alpha\in\Sigma^+}, \forall m \in \mathcal{U}_{\Sigma^+}$.

Recall that W acts on $\mathbf{M}$ by $(w.z)_{\alpha} = z_{w^{-1}\alpha}, \forall \alpha \in \Sigma, w \in W, z \in \mathbf{M}$, we obtain $\mathcal{U}_{w(\Sigma^+)} = w.(\mathcal{U}_{\Sigma^+}), \forall w \in W$. Then, it follows from [2, Lemma 1.2] that pair $(\mathcal{U}_{\Sigma^+}, \chi_{\Sigma^+})$ is a chart on $\mathbf{M}$ and $\{(\mathcal{U}_{w(\Sigma^+)}, \chi_{w(\Sigma^+)})\}_{w\in W}$ defines an atlas of charts on $\mathbf{M}$ such that $\mathbf{M}$ becomes a real analytic submanifold.

Now, for each $a \in A_{\mathbf{C}}$ and $\alpha \in \Sigma$, we define $a^{\alpha} = e^{\alpha \log a} \in \mathbf{C}^* = \mathbf{C} \setminus \{0\}$ and consider subset

$$A_{\mathrm{IR}} = \{ a \in A_{\mathbf{C}} \mid a^{\alpha} \in \mathrm{IR}, \forall \alpha \in \Sigma \}.$$

By definition, $\varphi(A_{\mathrm{IR}})$ is a subset of $(\mathrm{IRIP}^1)^{\Sigma}$. Denote the closure of $\varphi(A_{\mathrm{IR}})$ in $(\mathrm{IRIP}^1)^{\Sigma}$ by $\mathcal{A}_{\mathrm{IR}}$. It follows from (1) and (2) that $\mathcal{A}_{\mathrm{IR}}$ is a compact subset in $\mathbf{M}$.

Let $\mathcal{U}_{\Delta}$ be the subset of $\mathcal{U}_{\Sigma^+}$ consisting of elements $m = (m_{\alpha}, m_{-\alpha})$ such that

$$m_{\alpha} = \prod_{\gamma\in\Delta}(m_{\gamma})^{k(\alpha,\gamma)}, \forall \alpha \in \Sigma^+, \alpha = \sum_{\gamma\in\Delta} k(\alpha,\gamma).\gamma.$$

Then, $\mathcal{U}_{\Delta}$ is an open subset in $\mathcal{A}_{\mathrm{IR}}$. It follows that

$$\chi_{\Sigma^+}(\mathcal{U}_{\Delta}) = \{ x \in \mathrm{IR}^{\Sigma^+} \mid x_{\alpha} = \prod_{\gamma\in\Delta}(x_{\gamma})^{k(\alpha,\gamma)} \}$$

and we get homeomorphism $\chi_{\Delta} : \mathcal{U}_{\Delta} \to \mathrm{IR}^{\Delta}$ defined by $\chi_{\Delta}(m) = (m_{\gamma})_{\gamma\in\Delta}$, for all $m \in \mathcal{U}_{\Delta}$. In addition, we can define an atlas of charts on $\mathcal{A}_{\mathrm{IR}}$ induced from the atlas of charts that is defined on M.

Theorem 2.1 ([2, *Theorem 1.4*]) *$\overline{A}_{\mathrm{IR}}$ is a compact real analytic manifold that is called a compactification of A_{IR}. The set of charts $\{(\mathcal{U}_{w(\Delta)}, \chi_{w(\Delta)})\}_{w\in W}$ defines an atlas of charts on $\overline{A}_{\mathrm{IR}}$ so that the manifold $\overline{A}_{\mathrm{IR}}$ is covered by $|W|$-many charts.*

Example 2.2 Consider real semi-simple Lie group $G = SL(n,\mathrm{IR})$ and denote $\mathfrak{g} = sl(n,\mathrm{IR})$, the corresponding Lie algebra of G. Suppose that θ is the Cartan involution defined by $\theta(X) = ({}^t X)^{-1}, \forall X \in G$ and $K = SO(n)$ is the maximal compact subgroup in G with respect to θ. Then, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$ with respect to θ, where $\mathfrak{k} = so(n)$ is the Lie algebra of K. Moreover, we have that $G/K = SL(n,\mathrm{IR})/SO(n)$ is a Riemannian symmetric space of non-compact type.

Then, we get a maximal abelian subspace in $\mathfrak{g}$ defined by

$$\mathfrak{a} = \left\{ \begin{pmatrix} t_1 & 0 & \dots & 0 \\ 0 & t_2 & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & t_n \end{pmatrix} \mid t_1 + t_2 + \dots + t_n = 0 \right\}.$$

By definition, root system Σ of $\mathfrak{a}$ in $\mathfrak{g}$ is $\Sigma = \{ e_i - e_j \mid 1 \le i \ne j \le n \}$, and the Weyl group W is isomorphic to S_n, where S_n is the symmetric group of order n. Moreover, the corresponding analytic subgroup in G of $\mathfrak{a}$ is defined by

$$A = \left\{ \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & a_n \end{pmatrix} \mid a_1 a_2 \dots a_n = 1,\ a_i > 0 \right\} \simeq (0,\infty)^{n-1}.$$

Then, we get

$$A_{\mathrm{IR}} = \left\{ \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & a_n \end{pmatrix} \mid a_1 a_2 \dots a_n = 1 \right\} \simeq (\mathrm{IR}^*)^{n-1}.$$

By definition, we have

$$\mathbf{M} = \{ z \in (\mathrm{IRIP}^1)^{\Sigma} \mid z_{-\alpha} = z_\alpha^{-1}, \forall \alpha \in \Sigma \}$$

$$= \{ (z_\gamma, z_{-\gamma}) \mid z_\gamma \in \mathrm{IP}^1(\mathrm{IR}), \gamma \in \Sigma^+ \} \simeq (\mathrm{IRIP}^1)^{\Sigma^+}.$$

Moreover, $\mathcal{U}_{\Sigma^+} = \{ m = (m_\alpha, m_{-\alpha}) \in \mathbf{M} \mid m_\alpha \in \mathrm{IR}, \forall \alpha \in \Sigma^+ \} \simeq \mathrm{IR}^{\Sigma^+}$, where $|\Sigma^+| = \dfrac{n(n-1)}{2}$ and the corresponding homeomorphism $\chi_{\Sigma^+} : \mathcal{U}_{\Sigma^+} \to \mathrm{IR}^{\Sigma^+}$ is defined by

$$\chi_{\Sigma^+}(m) = (m_\alpha)_{\alpha\in\Sigma^+}, \forall m \in \mathcal{U}_{\Sigma^+}.$$

It follows that pair $(\mathcal{U}_{\Sigma^+}, \chi_{\Sigma^+})$ is a chart on $\mathbf{M}$, and $\{(\mathcal{U}_{w(\Sigma^+)}, \chi_{w(\Sigma^+)})\}_{w\in W}$ defines an atlas of charts on $\mathbf{M}$ such that compact manifold $\mathbf{M}$ is covered by $n!$-many charts. By definition, we see that

$$\mathcal{U}_\Delta = \{ m\in \mathbf{M} \mid m_\alpha = \prod_{\gamma\in\Delta}(m_\gamma)^{k(\alpha,\gamma)},\ \alpha = \sum_{\gamma\in\Delta} k(\alpha,\gamma).\gamma,\ \forall \alpha\in\Sigma^+\}$$

$$\chi_{\Sigma^+}(\mathcal{U}_\Delta) = \{ x\in \mathrm{IR}^{\Sigma^+} \mid x_\alpha = \prod_{\gamma\in\Delta}(x_\gamma)^{k(\alpha,\gamma)} \} \simeq \mathrm{IR}^{\Delta} \simeq \mathrm{IR}^{n-1}.$$

Then, we get homeomorphism $\chi_\Delta : \mathcal{U}_\Delta \to \mathrm{IR}^{n-1}$ defined by

$$\chi_\Delta(m) = (m_\gamma)_{\gamma\in\Delta},\ \forall m\in \mathcal{U}_\Delta.$$

Hence, $A_{\mathrm{IR}} \simeq \mathrm{IR}^{n-1}\cup\{\infty\} \simeq S^{n-1}$ is a compact real analytic manifold and set of charts $\{\mathcal{U}_{w(\Delta)}, \chi_{w(\Delta)}\}_{w\in W}$ defines an atlas of charts on A_{IR} so that manifold A_{IR} is covered by $n!$-many charts.

3 A realization of Riemannian symmetric spaces

Consider subset $\bar{A}_{\mathrm{IR}} = \{ \tilde{a}\in A_{\mathrm{IR}} \mid (\tilde{a})^\alpha \in [-1,1], \forall \alpha\in\Sigma \}$ and recall that the Weyl group W acts on A_{IR} by $(w.\tilde{a})_\alpha = (\tilde{a})_{w^{-1}\alpha}, \forall w\in W, \forall \tilde{a}\in A_{\mathrm{IR}}$. Note that A_{IR} acts naturally on A_{IR}. Then, by definition, for each $\tilde{a}\in \bar{A}_{\mathrm{IR}}$, there exist $t\in[-1,1]^\Delta$ and $a_t\in A_{\mathrm{IR}}$ such that $\tilde{a} = a_t.sgn\,t$ and this decomposition is unique. Here, $sgn\,t = (sgn\,t_\gamma)_{\gamma\in\Delta}$ and for an s in IR, we define $sgn\,s = 1$ (resp. $0,-1$) if $s>0$ (resp. $s=0, s<0$).

Now, for each $\tilde{a}\in A_{\mathrm{IR}}$, there exists $w\in W$ such that $\tilde{a}\in \mathcal{U}_{w(\Sigma^+)} = w.(\mathcal{U}_{\Sigma^+})$. By choosing a suitable positive system Σ^+, we obtain $W.\bar{A}_{\mathrm{IR}} = A_{\mathrm{IR}}$.

Based on this, for each $\tilde{a}\in A_{\mathrm{IR}}$, we have unique decomposition $\tilde{a} = \tilde{a}_{fin}.\varepsilon(\tilde{a})$, where $\tilde{a}_{fin}\in A_{\mathrm{IR}}$ and $\varepsilon(\tilde{a})\in A_{\mathrm{IR}}$ such that $\varepsilon(\tilde{a})^\gamma \in \{-1,0,+1,\infty\}, \forall\gamma\in\Delta$.

Motivation for the Oshima's definition [5], $\varepsilon(\tilde{a})$ is called an extended signature of roots with respect to element $\tilde{a}$.

Note that for $\tilde{a}\in \bar{A}_{\mathrm{IR}}$, we obtain $\varepsilon(\tilde{a})\in\{-1,0,+1\}^\Delta$ and for all $\alpha\in\Sigma$, $\alpha = \sum_{\gamma\in\Delta} k(\alpha,\gamma).\gamma$, we have

$$\varepsilon(\tilde{a})^\alpha = \prod_{\gamma\in\Delta}(\varepsilon(\tilde{a})^\gamma)^{|k(\alpha,\gamma)|}.$$

It follows that mapping $\varepsilon_{\tilde{a}}$ of Σ to $\{-1,0,+1\}$ is defined by

$$\varepsilon_{\tilde{a}} : \Sigma\to\{-1,0,+1\},\ \alpha\mapsto \varepsilon(\tilde{a})^\alpha$$

is an extended signature of roots that is defined in [7, Definition 2.1].

Now, we go to define parabolic subalgebras with respect to extended signatures $\varepsilon(\tilde{a})$, for all $\tilde{a} \in A_{\mathrm{IR}}$.

First, we consider $\tilde{a} \in \bar{A}_{\mathrm{IR}}$ and let $\varepsilon = \varepsilon(\tilde{a})$ denotethe corresponding extended signature of roots. Put $F_\varepsilon = \{ \gamma \in \Delta \mid \varepsilon_{\tilde{a}}(\gamma) = \varepsilon(\tilde{a})^{\gamma} \neq 0 \}$ and $\Sigma_{F_\varepsilon} = (\sum_{\gamma \in F_\varepsilon} \mathrm{IR}\gamma) \cap \Sigma$. Let $\Sigma^+_{F_\varepsilon} = \Sigma^+ \cap \Sigma_{F_\varepsilon}$. Then, (see [8]) following subsets

$$\mathfrak{a}_\varepsilon = \{H \in \mathfrak{a} \mid \alpha(H) = 0, \text{for any } \alpha \in F_\varepsilon\},$$

$$\mathfrak{a}(\varepsilon) = \{H \in \mathfrak{a} \mid < H, H' >= 0, \text{for any } H' \in \mathfrak{a}_\varepsilon\},$$

$$\mathfrak{n}_\varepsilon = \sum_{\alpha \in \Sigma^+ - \Sigma^+_\varepsilon} \mathfrak{g}_\alpha,\ \mathfrak{n}^-_\varepsilon = \theta(\mathfrak{n}_\varepsilon),$$

$$\mathfrak{n}(\varepsilon) = \sum_{\alpha \in \Sigma^+_\varepsilon} \mathfrak{g}_\alpha,\ \mathfrak{n}^-(\varepsilon) = \theta(\mathfrak{n}(\varepsilon)),$$

$$\mathfrak{m}_\varepsilon = \mathfrak{m} + \mathfrak{n}(\varepsilon) + \mathfrak{n}^-(\varepsilon) + \mathfrak{a}(\varepsilon)$$

are Lie subalgebras of $\mathfrak{g}$.

Let W_{F_ε} be the subgroup of W generated by reflections with respect to γ in F_ε and let $A_\varepsilon, A(\varepsilon), N_\varepsilon, N^-_\varepsilon, N(\varepsilon), N^-(\varepsilon)$ and $(M_\varepsilon)_0$ denote the analytic subgroups of G to $\mathfrak{a}_\varepsilon, \mathfrak{a}(\varepsilon), \mathfrak{n}_\varepsilon, \mathfrak{n}^-_\varepsilon, \mathfrak{n}(\varepsilon), \mathfrak{n}^-(\varepsilon)$ and $\mathfrak{m}_\varepsilon$, respectively.

Then, we can define parabolic subalgebra $\mathfrak{p}_\varepsilon$ in $\mathfrak{g}$, where $\mathfrak{p}_\varepsilon = \mathfrak{m}_\varepsilon + \mathfrak{a}_\varepsilon + \mathfrak{n}_\varepsilon$ is its Langlands decomposition. Let P_ε denotethe parabolic subgroup in G with respect to $\mathfrak{p}_\varepsilon$, we see that $P_\varepsilon = M_\varepsilon A_\varepsilon N_\varepsilon$ is the corresponding Langlands decomposition of P_ε.

Moreover, it follows from [7, Lemma 2.3] that $P(\varepsilon) = (M_\varepsilon \cap K)A_\varepsilon N_\varepsilon$ is a closed subgroup of G, where $M_\varepsilon = (M_\varepsilon)_0 M$ and

$$N^- \times A(\varepsilon) \times P(\varepsilon) \to G, (n, a, p) \mapsto nap$$

is an analytic diffeomorphism onto an open submanifold of G.

In general, for each $\tilde{\eta} = w.\tilde{a} \in A_{\mathrm{IR}}$, where $w \in W$ and $\tilde{a} \in \bar{A}_{\mathrm{IR}}$, we first consider parabolic subgroup $P_\varepsilon = M_\varepsilon A_\varepsilon N_\varepsilon$ with respect to $\varepsilon = \varepsilon(\tilde{a})$, the corresponding extended signature of $\tilde{a}$. Then, we can define parabolic subgroup $P_{\tilde{\eta}} = \underline{w}.P_\varepsilon.\underline{w}^{-1}$ based on the action of the Weyl group W on parabolic subgroup P_ε. Here, $\underline{w}$ denotes a representative for $w \in W$ in $N_K(\mathfrak{a})$ [1].

Now, we put $\Sigma' = \{\alpha \in \Sigma \mid 2\alpha \notin \Sigma; \frac{\alpha}{2} \notin \Sigma\}$, and denote $\Sigma'_\varepsilon = \{\alpha \in \Sigma' \mid \varepsilon_{\tilde{a}}(\alpha) = 1\}$ for every extended signature $\varepsilon_{\tilde{a}}$ of roots defined by $\varepsilon(\tilde{a})$. Then, (see [7]) it follows that Σ' and Σ'_ε

are reduced root systems. Let W', W'_ε and W'_{F_ε} be the subgroups of W generated by the reflections with respect to the roots in $\Sigma', \Sigma'_\varepsilon$ and Σ'_{F_ε}.

Put $A'_{\mathrm{IR}} = W'.\bar{A}_{\mathrm{IR}}$ and consider product manifold $G \times A'_{\mathrm{IR}}$. Let $x = (g, \tilde{\eta})$ be an element of $G \times A_{\mathrm{IR}}$, where $\tilde{\eta} = w.\tilde{a}$, in which $w \in W'$ and $\tilde{a} \in \bar{A}_{\mathrm{IR}}$. Then, we get $\varepsilon_x = \varepsilon(\tilde{a})$, the extended signature of roots with respect to $\tilde{a}$. For simplicity, we use letters $P(x), F_x, \Sigma_x, \Sigma'_x, W'_x, \ldots$ instead of $P(\varepsilon_x), F_{\varepsilon_x}, \Sigma_{\varepsilon_x}, \Sigma'_{\varepsilon_x}, W'_{F_{\varepsilon_x}}, \ldots$, respectively.

Let $\{H_1, H_2, \ldots, H_l\}$ denote the dual basis of $\Delta = \{\alpha_1, \ldots, \alpha_l\}$, that is, $H_j \in \mathfrak{a}$ and $\alpha_i(H_j) = \delta_{ij}, \forall i, j = 1, 2, \ldots, l$ and put $a(x) = exp(-\frac{1}{2}\sum_{\gamma \in F_x} log\, |\, t_\gamma \,|\, H_\gamma)$, where $H_\gamma \in \{H_1, H_2, \ldots, H_l\}$ with respect to γ.

Let $W(x) = \{w \in W_x \mid \Sigma_x \cap w\Sigma^+ = \Sigma_x \cap \Sigma^+\}$. By [7, Lemma 2.5], we see that $W(x) = \{w \in W'_x \mid \Sigma'_x \cap w\Sigma^+ = \Sigma'_x \cap \Sigma^+\}$.

Now, we go to define an equivalent relation for points in $G \times A'_{\mathrm{IR}}$.

Definition 3.1 *We say that two elements* $x = (g, \omega.\tilde{a})$ *and* $x' = (g', \omega'.\tilde{a})$ *of* $G \times A'_{\mathrm{IR}}$ *are equivalent if and only if the following conditions hold*

(i) $w.\varepsilon_x = w'.\varepsilon_{x'}$

(ii) $w^{-1}w' \in W(x)$

(ii) $ga(x)P(x)\underline{w} = g'a(x')P(x)\underline{w'}$.

Then, it follows that (see [7]) Definition 3.1 really gives an equivalence relation, which we write $x \sim x'$. Moreover, assume that two points x and x' in $G \times A'_{\mathrm{IR}}$ satisfy conditions (i) and (ii), we get that $Ad(\underline{w'}^{-1}\underline{w})(\mathfrak{p}(x)) = \mathfrak{p}(x')$, where $\mathfrak{p}(x)$ and $\mathfrak{p}(x')$ are Lie algebras of Lie groups $P(x)$ and $P(x')$, respectively. Here, we note that Lie algebra $\mathfrak{p}(x)$ of Lie group $P(x)$ has the following form

$$\mathfrak{p}(x) = \mathfrak{m} + \mathfrak{a}_x + \sum_{\alpha \in \Sigma}\{X + \varepsilon_{\tilde{a}}(\alpha)\theta(X) \mid X \in \mathfrak{g}_\alpha\}.$$

Based on this and the relation $\underline{w}M\underline{w}^{-1} = \underline{w'}M\underline{w'}^{-1}$, we have $\underline{w}P(x)\underline{w}^{-1} = \underline{w'}P(x')\underline{w'}^{-1}$. It follows that the condition (iii) is equivalent to

$$ga(x)P(x) = g'a(x')\underline{w'}^{-1}\underline{w}P(x) \quad (3)$$

in $G / P(x)$.

Then, we see that the action of G on $G\times A'_{\mathrm{IR}}$ is compatible with the equivalence relation. The quotient space of $G\times A'_{\mathrm{IR}}$ by this equivalence relation then becomes a topological space with the quotient topology and denoted by $\mathbf{X}'$.

Let $\pi: G\times A'_{\mathrm{IR}} \to \mathbf{X}'$ be the natural projection. Since the action of G on $G\times A'_{\mathrm{IR}}$ is compatible with the equivalence relation, we can define an action of G on $\mathbf{X}'$ by

$$g_1\pi(g,\tilde{a}) = \pi(g_1 g,\tilde{a}), \forall g, g_1 \in G, \tilde{a}\in A'_{\mathrm{IR}}. \quad (4)$$

Put $A'_{\mathrm{IR},\varepsilon} = \{\tilde{a}\in A'_{\mathrm{IR}} \mid \varepsilon(\tilde{a}) = \varepsilon\}$ for each $\varepsilon\in\{-1,0,1\}^{\Delta}$ and $\mathbf{X}'_{\varepsilon} = \pi(G\times A'_{\mathrm{IR},\varepsilon})$.

Proposition 3.2 *The quotient topological space* $\mathbf{X}'$ *has the following properties:*

(i) $\mathbf{X}'$ *is a compact connected* G *-space and* $\mathbf{X}' = \bigcup_{\varepsilon\in\{-1,0,1\}^{\Delta}} \mathbf{X}'_{\varepsilon}$ *gives the orbital decomposition of* $\mathbf{X}'$ *for the action of* G *on it.*

(ii) $\mathbf{X}'_{\varepsilon} = \pi(G\times A'_{\mathrm{IR},\varepsilon})$ *is homeomorphic to* $G/P(\varepsilon)$ *for each* $\varepsilon\in\{-1,0,1\}^{\Delta}$.

Proof. (i) Since $\pi(G\times w.\mathcal{U}_{\Delta})$ is connected for every $w\in W'$, and W' is generated by elements $w_{\beta_1},...,w_{\beta_{l'}}$, where $\{\beta_1,...,\beta_{l'}\}$ is a fundamental system of roots for Σ', we see that the quotient space $\mathbf{X}'$ is connected.

Consider compact subset $K\times A'_{\mathrm{IR}} = K\times W'.\bar{A}_{\mathrm{IR}} \simeq K\times[-1,1]^{\Delta}\times W'$ of the product manifold $G\times A'_{\mathrm{IR}}$. Then, subset $\pi(K\times A_{\mathrm{IR}})$ is also compact because it is the image of a compact set under continuous map π.

Let $\mathfrak{a}^+$ denote the positive chamber corresponding to Σ^+ and put $A^+ = \exp\mathfrak{a}^+$. Let $\overline{A^+} = \{\exp X \mid X\in\mathfrak{a} \text{ with } \alpha(X)\geq 0 \text{ for all } \alpha\in\Sigma^+\}$ denote the closure of A^+, we see that

$$\overline{A^+} = \{\exp(-\frac{1}{2}\sum_{\gamma}(\log t_\gamma)H_\gamma) \mid t_\gamma\in(0,1]\}.$$

Fix $\varepsilon\in\{-1,1\}^{\Delta}$ and let $x=(g,\omega.\tilde{a})$ be an arbitrary point in $G\times A'_{\mathrm{IR},\varepsilon}$. Then, it follows from Cartan decomposition $G = K\overline{A^+}K$ [8] that there exist $k\in K$ and $h\in\overline{A^+_x}$ such that

$$khP(x) = ga(x)P(x),$$

where $\overline{A^+_x} = \{\exp X \mid X\in\mathfrak{a} \text{ with } \alpha(X)\geq 0 \text{ for all } \alpha\in\Sigma^+_x\}$.

Note that $\mathfrak{a}^+_x$ is a fundamental domain for the action of W_x, so we can apply Lemma 2.5 in [7] to imply that compact set $\pi(K\times A'_{\mathrm{IR}})$ contains subset $\pi(G\times A'_{\mathrm{IR},\varepsilon})$ for every $\varepsilon\in\{-1,1\}^{\Delta}$.

Moreover, since $G\times A'_{\mathrm{IR},\varepsilon}$ is dense in $G\times A'_{\mathrm{IR}}$ and $K\times A'_{\mathrm{IR}}$ is compact, it follows that $\pi(G\times A'_{\mathrm{IR},\varepsilon})$ is also dense in $\mathbf{X}'$. Then, $\mathbf{X}'$ is also compact.

(ii) Put $\tilde{a}\in A_{\mathrm{IR},\varepsilon}$ for each $\varepsilon\in\{-1,0,1\}^{\Delta}$ and define a map $\Psi: G/P(\varepsilon)\to \mathbf{X}_{\varepsilon}$ by $\Psi(gP(\varepsilon))=\pi(g,\tilde{a}), \forall g\in G.$ Then, we we can prove that the map is well defined and becomes an homeomorphism which is equivariant for the action of G. This follows the Proposition.

Now, we construct an analytic structure on topological space $\mathbf{X}'$ based on the analytic structure on A_{IR}.

Consider the atlas of charts $\{(\mathcal{U}_{w(\Delta)},\chi_{w(\Delta)})\}_{w\in W}$ on A_{IR} defined in Theorem 2.1, where $\mathcal{U}_{w(\Delta)}=w.\mathcal{U}_{\Delta}$, and $\chi_{w(\Delta)}:\mathcal{U}_{w(\Delta)}\to \mathrm{IR}^{w(\Delta)}$ is a homeomorphism defined by

$$\chi_{w(\Delta)}(w.m)=(m_{w^{-1}.\gamma})_{\gamma\in\Delta},\ \forall m\in\mathcal{U}_{\Delta}, w\in W.$$

For every $g\in G$ and $w\in W'$, we put $\Omega_g^w=\pi(gN^-\times\mathcal{U}_{w(\Delta)})$, in which N^- is the analytic subgroup of G corresponding to $\mathfrak{n}^-=\theta(\mathfrak{n})$, where $\mathfrak{n}=\sum_{\alpha\in\Sigma^+}\mathfrak{g}_\alpha$ and define $\Phi_g^w: N^-\times\mathrm{IR}^{\Delta}\to\Omega_g^w$ by $\Phi_g^w(n,t)=\pi(gn,w.\tilde{a}_t), \forall (n,t)\in N^-\times\mathrm{IR}^{\Delta}$.

Based on homeomorphism $\chi_{w(\Delta)}:\mathcal{U}_{w(\Delta)}\to\mathrm{IR}^{w(\Delta)}$ with respect to $w\in W'$, we can define a homeomorphism between $gN^-\times\mathcal{U}_{w(\Delta)}$ and $gN^-\times\{w\}\times\mathrm{IR}^{|\Delta|}$ for every $g\in G$. Combine this with the proof of Lemma 2.8 (ii) in [7], we get

Lemma 3.3 *For every* $g\in G$ *and* $w\in W'$, Φ_g^w *is a homeomorphism of* $N^-\times\mathrm{IR}^{\Delta}$ *onto an open subset* $\Omega_g^w=\pi(gN^-\times\mathcal{U}_{w(\Delta)})$ *of* $\mathbf{X}'$.

Now, consider the map $(\Phi_{g'}^{w'})^{-1}\circ(\Phi_g^w):(\Phi_g^w)^{-1}(\Omega_g^w\cap\Omega_{g'}^{w'})\to(\Phi_{g'}^{w'})^{-1}(\Omega_g^w\cap\Omega_{g'}^{w'})$ for $g,g'\in G$ and $w,w'\in W'$. By definition, Φ_g^w is bijective and continuous. It follows that $(\Phi_{g'}^{w'})^{-1}\circ(\Phi_g^w)$ is bijective and its inverse is of the same form. Moreover, we have

Lemma 3.4 *Let* $g,g'\in G$ *and* $w,w'\in W'$. *Then,*

$$(\Phi_{g'}^{w'})^{-1}\circ(\Phi_g^w):(\Phi_g^w)^{-1}(\Omega_g^w\cap\Omega_{g'}^{w'})\to(\Phi_{g'}^{w'})^{-1}(\Omega_g^w\cap\Omega_{g'}^{w'})$$

defines an analytic diffeomorphism between the open subsets of $N^-\times\mathrm{IR}^{\Delta}$.

Proof. We have only to show that $(\Phi_{g'}^{w'})^{-1}\circ(\Phi_g^w)$ is analytic.

Note that based on the homeomorphism between $gN^-\times\mathcal{U}_{\Delta}$ and $gN^-\times\mathrm{IR}^{|\Delta|}$ for every $g\in G$, we can prove the Lemma under condition $w=w'$ by the same way as the proof of Lemma 7 in [5]. Now, we will prove the Lemma without condition $w'=w$.

For any $q\in\Omega_g^w\cap\Omega_{g'}^{w'}$, we can choose (n,t) and (n',t') in $N^-\times\mathrm{IR}^{\Delta}$ such that $\pi(x)=\pi(x')=q$, where $x=(gn,w.\tilde{a}_t)$ and $x'=(g'n',w'.\tilde{a}_{t'})$. Then, we have

$$gna(x)P(x) = g'n'a(x')\underline{w'}^{-1}\underline{w}P(x)$$

by (3). Put $g_1 = gna(x)$, $g_2 = g'n'a(x')$, $g_3 = g_1\underline{w}^{-1}\underline{w'}$ and consider maps

$$(\Phi_{g'}^{w'})^{-1} \circ \Phi_{g_2}^{w'} : (n,t) \mapsto (e,\varepsilon_t), (\Phi_{g_2}^{w'})^{-1} \circ \Phi_{g_3}^{w'} : (e,\varepsilon_t) \mapsto (e,\varepsilon_{t'}),$$

$$(\Phi_{g_3}^{w'})^{-1} \circ \Phi_{g_1}^{w} : (n,t) \mapsto (e,\varepsilon_{t'}) \text{and } (\Phi_{g_1}^{w})^{-1} \circ \Phi_{g}^{w} : (e,\varepsilon_{t'}) \mapsto (n',t')$$

where $\varepsilon_t = \varepsilon(\tilde{a}_t)$ and $\varepsilon_{t'} = \varepsilon(\tilde{a}_{t'})$ belong to $\{-1,0,1\}^{\Delta}$.

Then, we see that $(\Phi_{g'}^{w'})^{-1} \circ (\Phi_{g}^{w})$ is the composition of the above maps

$$(\Phi_{g'}^{w'})^{-1} \circ (\Phi_{g}^{w}) = ((\Phi_{g'}^{w'})^{-1} \circ \Phi_{g_2}^{w'}) \circ ((\Phi_{g_2}^{w'})^{-1} \circ \Phi_{g_3}^{w'}) \circ ((\Phi_{g_3}^{w'})^{-1} \circ \Phi_{g_1}^{w}) \circ ((\Phi_{g_1}^{w})^{-1} \circ \Phi_{g}^{w}).$$

It follows from what we have mentioned at the beginning of the proof, maps $(\Phi_{g_1}^{w})^{-1} \circ \Phi_{g}^{w}, (\Phi_{g_2}^{w'})^{-1} \circ \Phi_{g_3}^{w'}$ and $(\Phi_{g'}^{w'})^{-1} \circ \Phi_{g_2}^{w'}$ are analytic diffeomorphisms between the open subsets of $N^{-} \times \mathrm{IR}^{\Delta}$.

Moreover, by a similar way as the proof of Lemma 2.8 (iii) in [7], we can show that the map $(\Phi_{g_3}^{w'})^{-1} \circ \Phi_{g_1}^{w}$ is analytic on an open subset of $N^{-} \times \mathrm{IR}^{\Delta}$ containing (e,ε_t).

Since (n,t) is an arbitrary element in $(\Phi_{g}^{w})^{-1}(\Omega_{g}^{w} \cap \Omega_{g'}^{w'})$, we see that $(\Phi_{g'}^{w'})^{-1} \circ (\Phi_{g}^{w})$ is analytic and the set $(\Phi_{g}^{w})^{-1}(\Omega_{g}^{w} \cap \Omega_{g'}^{w'})$ is open in $N^{-} \times \mathrm{IR}^{\Delta}$. Because the inverse of the map $(\Phi_{g'}^{w'})^{-1} \circ (\Phi_{g}^{w})$ also has the same property, we have the Lemma.

Lemma 3.3 and Lemma 3.4 assure that we can define an analytic structure on $\mathbf{X}'$ through maps Φ_{g}^{w} so that they define analytic diffeomorphisms onto open subsets Ω_{g}^{w} of $\mathbf{X}'$ and the action of G on $\mathbf{X}'$ is analytic. On the other hand, based on the homeomorphism between $gN^{-} \times \mathcal{U}_{\Delta}$ and $gN^{-} \times \mathrm{IR}^{|\Delta|}$ for every $g \in G$ and by a similar way as the proof of Theorem 2.7 in [7], we can prove that topological space $\mathbf{X}'$ is Hausdorff. Moreover, for an element $w \in W'$, the unique G -orbit which is isomorphic to G/K (resp. G/P) is just $G\pi(e, w.\tilde{a}_{\varepsilon_1})$ (resp. $G\pi(e, w.\tilde{a}_{\varepsilon_0})$). Combining this with Proposition 3.2, we get

Theorem 3.5 *The quotient topological space* $\mathbf{X}'$ *has the following properties:*

(i) $\mathbf{X}'$ *is a compact connected real analytic manifold and* $\bigcup_{w \in W', g \in G} \Omega_{g}^{w}$ *is an open covering of* $\mathbf{X}'$ *such that maps* Φ_{g}^{w} *are real analytic diffeomorphisms.*

(ii) *The action of* G *on* $\mathbf{X}'$ *is analytic and orbit* $G\pi(x)$ *for point* x *in* $\mathbf{X}'$ *is isomorphic to homogeneous space* $G/P(x)$. *In particular, the number of* G *-orbits which are isomorphic to* G/K *(resp.* G/P *) is just the number of elements in* W'.

Let $\mathrm{IR}_1^{\Delta} = \{ t \in \mathrm{IR}^{\Delta} \mid t_\alpha < 1 \text{ for every } \alpha \in \Sigma \}$. Note that for any elements $g \in G, k \in K, m \in M$ and $w.\tilde{a}_t \in A'_{\mathrm{IR}}$, we have $(gkm, w.\tilde{a}_t) \sim (gk, w.\tilde{a}_t)$. Then, we can define

$$\Psi_g^w : K/M \times \mathrm{IR}_1^{\Delta} \to \mathbf{X}', (kM, t) \mapsto \pi(gk, w.\tilde{a}_t).$$

Based on Lemma 2.9 in [7] and Theorem 3.5, we have

Corollary 3.6 $\Psi_g^w : (kM, t) \mapsto \pi(gk, w.\tilde{a}_t)$ *is an analytic diffeomorphism of product manifold* $K/M \times \mathrm{IR}_1^{\Delta}$ *to* $\mathbf{X}'$. *Moreover,* $\bigcup_{w \in W', g \in G} \mathrm{Im}\Psi_g^w$ *is an open covering of* $\mathbf{X}'$.

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