Hue University Journal of Science: Natural Science
Vol. 130, No. 1D, 75–82, 2021

pISSN 1859-1388
eISSN 2615-9678

WEAK LEFSCHETZ PROPERTY OF GRADED GORENSTEIN ALGEBRAS ASSOCIATED TO THE APÉRY SET OF A NUMERICAL SEMIGROUP

Tran Quang Hoa[1], Ho Vu Ngoc Phuong[2]*

[1] University of Education, Hue University, 34 Le Loi St., Hue, Vietnam
[2] University of Sciences, Hue University, 77 Nguyen Hue St., Hue, Vietnam

* Corresponding author: Ho Vu Ngoc Phuong <hvnphuong@husc.edu.vn>

(*Received: 28 June 2021; Accepted: 9 November 2021*)

Abstract. It has been conjectured that *all* graded Artinian Gorenstein algebras of codimension three have the weak Lefschetz property over a field of characteristic zero. In this paper, we study the weak Lefschetz property of associated graded algebras A of the Apéry set of M-pure symmetric numerical semigroups generated by four natural numbers. These algebras are graded Artinian Gorenstein algebras of codimension three.

Keywords: Apéry set, Artinian Gorenstein algebras, numerical semigroups, weak Lefschetz property

1 Introduction

The weak Lefschetz property (WLP for short) for an Artinian graded algebra A over a field K simply says that the multiplication by a general linear form $\times L:[A]_i \to [A]_{i+1}$ has a maximal rank in every degree i. At first glance, this might seem to be a simple problem of linear algebra. However, determining which graded Artinian K-algebras have the WLP is notoriously difficult. Most authors have studied the problem from different points of view, applying tools from representation theory, topology, vector bundle theory, plane partitions, differential geometry, among others (see for instance [1-9]).

One of the most interesting open problems in this field is whether *all* graded Artinian Gorenstein algebras of codimension three have the WLP in characteristic zero. In the special case of codimension three complete intersections, a positive answer was obtained in characteristic zero in [10] by using the Grauert-Mülich theorem. For positive characteristic, however, only the case of monomial complete intersections has been studied (see [11, 12]), applying different approaches from combinatorics.

For the case of Gorenstein algebras of codimension three that are not necessarily complete intersections, it is known that for each possible Hilbert function, an example exists, having the WLP [13]. Some partial results are given in [14] to show that for certain Hilbert functions, all such Gorenstion algebras have the WLP. It was shown in [15] that all codimension three Artinian Gorenstein algebras of socle degree at most 6 have the WLP in characteristic zero, but the general case remains completely open.

In this work, we study whether Artinian Gorenstein algebras of codimension three associated to the Apéry set of numerical semigroups have the WLP. More precisely, we consider a numerical semigroup P generated by $\{a_1, a_2, a_3, a_4\} \subset \mathbb{N}^4$ with $\gcd(a_1, a_2, a_3, a_4) = 1$. The *Apéry set* $Ap(P)$ *of* P with respect to the minimal generator of the semigroup is defined as follows:

DOI: 10.26459/hueunijns.v130i1D.5893

$$Ap(P) := \{a \in P \mid a - a_1 \notin P\} = \{0 = \omega_1 < \omega_2 < \cdots < \omega_{a_1}\}.$$

Notice that $Ap(P)$ is a finite set and $\#Ap(P) = a_1$. Recall that a numerical semigroup P is said to be M*-pure symmetric* if for each $i = 1, \ldots, a_1$, $\omega_i + \omega_{a_1 - i+1} = \omega_{a_1}$ and

$ord(\omega_i) + ord(\omega_{a_1-i+1}) = ord(\omega_{a_1})$, where

$$ord(a) := \max\{\sum_{i=1}^{4} \lambda_i \mid a = \sum_{i=1}^{4} \lambda_i a_i\}$$

is the *order* of $a \in P$. Therefore, the Apéry set of a M-pure symmetric semigroup has the structure of a symmetric lattice.

Let K be a field of characteristic zero and consider the homomorphism

$$\Phi : S := K[x_1, \ldots, x_4] \to K[P] := K[t^{a_1}, \ldots, t^{a_4}],$$

which sends $x_i \mapsto t^{a_i}$. Then, $K[P] \cong S / Ker(\Phi)$ is a one-dimensional ring associated to P. Now, set $\overline{S} = S/(x_1)$. Then, there is one to one correspondence between the elements of $Ap(P)$ and the generators of $\overline{S}$ as a K-vector space. Let $\overline{\mathfrak{m}}$ be the maximal homogeneous ideal of $\overline{S}$, define the *associated graded algebra* of the Apéry set of P

$$A = gr_{\overline{\mathfrak{m}}}(\overline{S}) := \bigoplus_{i \geq 0} \frac{\overline{\mathfrak{m}}^i}{\overline{\mathfrak{m}}^{i+1}}.$$

It follows that A is a standard graded Artinian K-algebra of codimension three. In his work [16], Bryant proved that A is Gorenstein if and only if P is M-pure symmetric. In a recent paper [17], Guerrieri showed that if A is an Artinian Gorenstein algebra that is not a complete intersection, then A is of form $A = R/I$ when $R = K[x, y, z]$ and

$$I = (x^a, y^b - x^{b-\gamma} z^{\gamma}, z^c, x^{a-b+\gamma} y^{b-\beta}, y^{b-\beta} z^{c-\gamma}), \quad (1.1)$$

where $1 \leq \beta \leq b-1$, $\max\{1, b-a+1\} \leq \gamma \leq \min\{b-1, c-1\}$ and $a \geq c \geq 2$. The integers a, b, c, β and γ are determined from the structure of $Ap(P)$ [17, Section 5].

Our goal is to study the WLP of the graded algebra A. Our main results are as follows:

Theorem. *Consider the ideal I as in (1.1). If $a \geq b + c - 2$, then R/I has the WLP for any $1 \leq \beta \leq \min\{a - b - c + 3, b - 1\}$. As a consequence, if $a \geq 2b + c - 4$, then R/I has the WLP.*

Then, we recover a result in [8, Theorem 3.7], with a shorter and easier proof.

Corollary. *Let I be as in (1.1). If one of the a, b and c is equal to two, then R/I has the WLP.*

2 Background and preparatory results

In this section, we fix the notations, and we recall the known facts needed later in this work. We fix K as a field of characteristic zero and $R = K[x_1, \ldots, x_n]$. Let

$$A = R/I = \bigoplus_{i=0}^{D} [A]_i$$

be a graded Artinian algebra.

Definition 2.1. *For any graded Artinian algebra $A = R/I = \bigoplus_{i=0}^{D} [A]_i$, the Hilbert function of A is the function*

$$h_A : \mathbb{N} \to \mathbb{N}$$

defined by $h_A(t) = \dim_K [A]_t$. As A is Artinian, its Hilbert function is equal to its *h-vector* that one can express as a sequence

$$\underline{h}_A = (1 = h_0, h_1, h_2, h_3, \ldots, h_D),$$

where $h_i = h_A(i) > 0$, and D is the last index with this property. The integer D is called the

pISSN 1859-1388
eISSN 2615-9678

socle degree of A. The h-vector $\underline{h}_A$ is said to be *symmetric* if $h_{D-i} = h_i$ for every $i = 0,1,\ldots,\lfloor\frac{D}{2}\rfloor$.

Definition 2.2. [18, Proposition~2.1] *A graded Artinian algebra A as above is Gorenstein if and only if $h_D = 1$ and the multiplication map*

$$[A]_i \times [A]_{D-i} \to [A]_D \cong K$$

is a perfect pairing for all $i = 0,1,\ldots,\lfloor\frac{D}{2}\rfloor$.

It follows that the h-vector of a graded Artinian Gorenstein is symmetric.

Definition 2.3. *A graded Artinian K-algebra A has the weak Lefschetz property, briefly, WLP, if there exists $L \in [A]_1$ such that the map $\times L : [A]_i \to [A]_{i+1}$ has a maximal rank for each i. We also say that a homogeneous ideal I has the WLP if R/I has the WLP.*

We can determine the WLP by considering the rank of the multiplication map by a general linear form in every degree. However, for a standard graded Artinian Gorenstein K-algebra, the WLP is determined by considering only the multiplication map in one degree.

Proposition 2.4. [4, Proposition 2.1] *Let A be a standard graded Artinian Gorenstein K-algebra with the socle degree D and $k := \lfloor\frac{D}{2}\rfloor$. Then, we have*

1. If D is odd, A has the WLP if and only if there is an element $L \in [A]_1$ such that the multiplication map $\times L : [A]_k \to [A]_{k+1}$ is an isomorphism.

2. If D is even, A has the WLP if and only if there is an element $L \in [A]_1$ such that the multiplication map $\times L : [A]_k \to [A]_{k+1}$ is surjective or equivalently the multiplication map $\times L : [A]_{k-1} \to [A]_k$ is injective.

We close this section by recalling a result of Guerrieri.

Proposition 2.5. [17, Theorem 2.1] *Assume that $G = \bigoplus_{i=0}^{D} [G]_i$ is a standard graded Artinian Gorenstein K-algebra with the socle degree D that has the WLP. If $\ell \in [G]_1$ is a linear element, then the quotient ring*

$$A = \frac{G}{(0 :_G \ell)}$$

is also a standard graded Artinian Gorenstein K-algebra. Assume that G and A have the same codimension and set $k := \lfloor\frac{D}{2}\rfloor$. Then A has also the WLP, whenever D is odd; or D is even and $\dim_K [G]_{k-1} = \dim_K [G]_k$.

3 The WLP for a class of Artinian Gorenstein algebras of codimension three

From now on, we study the WLP of the ideal in the following setting.

Setting 3.1. *Let $R = K[x,y,z]$ be the standard graded polynomial ring over a field K of characteristic zero and consider the ideal*

$$I = (x^a, y^b - x^{b-\gamma} z^\gamma, z^c, x^{a-b+\gamma} y^{b-\beta}, y^{b-\beta} z^{c-\gamma}) \subset R,$$

where $1 \le \beta \le b-1, \max\{1, b-a+1\} \le \gamma \le \min\{b-1, c-1\}$ *and* $a \ge c \ge 2$.

It is clear that $a \ge b - c + 2$. Firstly, we need the following result.

Proposition 3.2. [8, Proposition 3.1] *Let $I_\beta := I$ be as in Setting 3.1. Set $A_\beta = R / I_\beta$ and $G = R/a$, where $a := (x^a, y^b - x^{b-\gamma} z^\gamma, z^c)$. Then, one has:*

1. $I_\beta = a :_R y^\beta$. Therefore, A_β is an Artinian Gorenstein algebra of codimension three and its socle degree is $D = a+b+c-\beta-3$.

2. G is an Artinian complete intersection of codimension three, and hence it has the WLP. For all $2 \le \beta \le b-1$

$$A_1 = \frac{G}{(0:_G y)} \quad \text{and} \quad A_\beta = \frac{A_{\beta-1}}{(0:_{A_{\beta-1}} y)}.$$

3. The free resolution of A_β is

$$0 \to R(-a-b-c+\beta) \to \begin{matrix} R(-a-b+\beta) \\ \oplus \\ R(-a-c+\beta) \\ \oplus \\ R(-b-c+\beta) \\ \oplus \\ R(-a-\gamma) \\ \oplus \\ R(-b-c+\gamma) \end{matrix} \to \begin{matrix} R(-a) \\ \oplus \\ R(-b) \\ \oplus \\ R(-c) \\ \oplus \\ R(-a-\gamma+\beta) \\ \oplus \\ R(-b-c+\gamma+\beta) \end{matrix} \to R \to A_\beta \to 0.$$

Next, we study the Hilbert functions of these algebras.

Proposition 3.3. *With notations as in Proposition 3.2. Set $k := \lfloor \frac{a+b+c-3}{2} \rfloor$.*

If $a \ge b+c+1$, then $H_G(k) = H_G(k-1)$.

Proof. We first observe that as $a \ge b+c+1$, we have $k-a+1 \le 0, k-b \ge 0$ and $k-c \ge 0$. Notice that G is resolved by the Koszul complex, we get

$$H_G(k) - H_G(k-1) = b+c-k-1+\binom{k-b-c+1}{1}.$$

We now consider the following two cases.

Case 1. $a+b+c$ **is odd.** In this case, one has $k = \frac{a+b+c-3}{2}$ and

$$H_G(k) - H_G(k-1) = \frac{b+c-a+1}{2} + \binom{\frac{a-b-c-1}{2}}{1}.$$

First, if $a = b+c+1$, then

$$H_G(k) - H_G(k-1) = 0$$

and now if $a \ge b+c+3$, then $a-b-c-1 \ge 2$. It follows that

$$H_G(k) - H_G(k-1) = \frac{b+c-a+1}{2} + \frac{a-b-c-1}{2} = 0,$$

which completes the argument whenever $a+b+c$ is odd.

Case 2. $a+b+c$ **is even.** In this case, one has $k = \frac{a+b+c-4}{2}$ and

$$H_G(k) - H_G(k-1) = \frac{b+c-a+2}{2} + \binom{\frac{a-b-c-2}{2}}{1}.$$

If $a = b+c+2$, then

$$H_G(k) - H_G(k-1) = 0$$

and if $a \ge b+c+4$, then $a-b-c-2 \ge 2$. It follows that

$$H_G(k) - H_G(k-1) = \frac{b+c-a+2}{2} + \frac{a-b-c-2}{2} = 0,$$

which gives the claim for this case. Thus, we complete the proof.

pISSN 1859-1388
eISSN 2615-9678

Proposition 3.4. *Let* $I_\beta := I$ *be as in Setting 3.1. Set* $A_\beta = R/I_\beta$ *and* $k = \lfloor \frac{a+b+c-\beta-3}{2} \rfloor$. *If* $a \ge b+c$, *then*

$$H_{A_\beta}(k) = H_{A_\beta}(k-1),$$

provided $1 \le \beta \le a-b-c+1$.

Proof. Since we have the free resolution of A_β as in Proposition 3.2(iii), we get

$$H_{A_\beta}(k) - H_{A_\beta}(k-1) = \binom{k+1}{1} - \binom{k-a+1}{1} - \binom{k-b+1}{1} - \binom{k-c+1}{1} - \binom{k-a-\gamma+\beta+1}{1} - \binom{k-b-c+\beta+\gamma+1}{1}$$

$$+\binom{k-a-b+\beta+1}{1} + \binom{k-a-c+\beta+1}{1} + \binom{k-b-c+\beta+1}{1}$$

$$+\binom{k-a-\gamma+1}{1} + \binom{k-b-c+\gamma+1}{1} - \binom{k-a-b-c+\beta+1}{1}$$

with convention $\binom{n}{m} = 0$ if $n < m$. The condition $\beta \le a-b-c+1$ implies that $k-a+\beta+1 \le 0$. Therefore,

$$\begin{cases} k-a+1 \le 0 \\ k-a-\gamma+1 \le 0 \\ k-a-\gamma+\beta+1 \le 0 \\ k-a-b+\beta+1 \le 0 \\ k-a-c+\beta+1 \le 0 \\ k-a-b-c+\beta+1 \le 0. \end{cases}$$

On the other hand, we also have that $k-b \ge 0$ and $k-c \ge 0$ since $\beta \le a-b-c+1$. Thus,

$$H_{A_\beta}(k) - H_{A_\beta}(k-1) = b+c-k-1-\binom{k-b-c+\gamma+\beta+1}{1} + \binom{k-b-c+\beta+1}{1} + \binom{k-b-c+\gamma+1}{1}. \quad (3.1)$$

To prove the proposition, we consider the following two cases.

Case 1. $a+b+c-\beta$ **is odd.** In this case, $k = \frac{a+b+c-\beta-3}{2}$. By (3.1), we get that

$$H_{A_\beta}(k) - H_{A_\beta}(k-1) = \frac{b+c-a+\beta+1}{2} - \binom{\frac{a-b-c+\beta-1}{2}+\gamma}{1} + \binom{\frac{a-b-c+\beta-1}{2}}{1} + \binom{\frac{a-b-c-\beta-1}{2}+\gamma}{1}$$

$$= b+c-a+1-\gamma + \binom{\frac{a-b-c+\beta-1}{2}}{1} + \binom{\frac{a-b-c-\beta-1}{2}+\gamma}{1}$$

as $a-b-c+\beta-1 \ge 0$ under the condition $1 \le \beta \le a-b-c+1$. Therefore, if $\beta = a-b-c+1$, then

$$H_{A_\beta}(k) - H_{A_\beta}(k-1) = b+c-a+1-\gamma + \binom{\beta-1}{1} + \binom{\gamma-1}{1}$$

$$= 2-\beta-\gamma + \binom{\beta-1}{1} + \binom{\gamma-1}{1} = 0.$$

If $\beta \le a-b-c-1$, then $a \ge b+c+2$, and hence

$$H_{A_\beta}(k) - H_{A_\beta}(k-1) = b+c-a+1-\gamma + \frac{a-b-c+\beta-1}{2} + \frac{a-b-c-\beta-1}{2} + \gamma = 0.$$

Case 2. $a+b+c-\beta$ **is even.** In this case, $1 \le \beta \le a-b-c$ and $k = \frac{a+b+c-\beta-4}{2}$. By (3.1), we get that

$$H_{A_\beta}(k) - H_{A_\beta}(k-1) = \frac{b+c-a+\beta+2}{2} - \binom{\frac{a-b-c+\beta-2}{2}+\gamma}{1} + \binom{\frac{a-b-c+\beta-2}{2}}{1} + \binom{\frac{a-b-c-\beta-2}{2}+\gamma}{1}$$

$$= b+c-a+2-\gamma+\binom{\frac{a-b-c+\beta-2}{2}}{1}+\binom{\frac{a-b-c-\beta-2}{2}+\gamma}{1}$$

as $a-b-c+\beta-2\geq 0$ under the condition $1\leq\beta\leq a-b-c$. Therefore, if $\beta=a-b-c$, then

$$H_{A_\beta}(k)-H_{A_\beta}(k-1)=b+c-a+2-\gamma+\binom{\beta-1}{1}+\binom{\gamma-1}{1}$$
$$=2-\beta-\gamma+\binom{\beta-1}{1}+\binom{\gamma-1}{1}=0.$$

If $\beta\leq a-b-c-2$, then $a\geq b+c+3$, and hence,

$$H_{A_\beta}(k)-H_{A_\beta}(k-1)=b+c-a+2-\gamma+\frac{a-b-c+\beta-2}{2}+\frac{a-b-c-\beta-2}{2}+\gamma=0.$$

Thus, the proposition is completely proved.

We now state our main result.

Theorem 3.5. *Let I be as in Setting 3.1. If $a\geq b+c-2$, then R/I has the WLP for any $1\leq\beta\leq\min\{a-b-c+3,b-1\}$. As a consequence, if $a\geq 2b+c-4$, then R/I has the WLP.*

Proof. Notice that the socle degree of $A_{a-b-c+2}$ is odd; hence, by applying Propositions 2.5 and 3.2 and Lemma 3.4, it is enough to show that A_1 has the WLP. We consider the following cases:

Case 1. $a+b+c$ is even. Then, the socle degree of G is odd. Since G is an Artinian complete intersection algebra of codimension 3, G has the WLP [10, Corollary 2.4]. Thus, it follows that A_1 has the WLP by Proposition 2.5(i).

Case 2. $a+b+c$ is odd. If $a-b-c\geq 1$, then A_1 has the WLP by Proposition 2.5 and Lemma 3.3. As $a+b+c$ is odd, we only consider the case where $a=b+c-1$. It remains to show that the ideal

$$I=(x^{b+c-1},y^b-x^{b-\gamma}z^\gamma,z^c,x^{c-1+\gamma}y^{b-1},y^{b-1}z^{c-\gamma})$$

has the WLP, where $1\leq\gamma\leq\min\{b-1,c-1\}$. Note that the socle degree of R/I is $2b+2c-5$, and hence $k=b+c-3$. Let $L=x-y-z$. By Proposition 2.4, it is enough to show that the multiplication

$$\times L:[R/I]_{b+c-3}\to[R/I]_{b+c-2}$$

is surjective, or equivalently $[R/(I,L)]_{b+c-2}=0.$ We have that

$$R/(I,L)\cong K[y,z]/J$$

where $J=((y+z)^{b+c-1},y^b-(y+z)^{b-\gamma}z^\gamma,z^c,(y+z)^{c-1+\gamma}y^{b-1},y^{b-1}z^{c-\gamma})$.

We will prove that $[K[y,z]/J]_{b+c-2}=0$, or equivalently $y^{b+c-2-i}z^i\in J$ for all $0\leq i\leq b+c-2.$ We do it by the induction on i.

We first observe that $z^c\in J$ and $c\leq b+c-2$; therefore, for any $0\leq i\leq b-2$

$$y^iz^{b+c-2-i}=y^iz^cz^{b-2-i}\in J.$$

Similarly, as $y^{b-1}z^{c-\gamma}\in J$, hence for every $b-1\leq i\leq b+\gamma-2$

$$y^iz^{b+c-2-i}=y^{b-1}z^{c-\gamma}y^{i-b+1}z^{b+\gamma-2-i}\in J.$$

It follows that $y^iz^{b+c-2-i}$ for any $0\leq i\leq b+\gamma-2.$ Finally, for any

$b+\gamma-1\leq i\leq b+c-2$, $i\geq b$,

and since $y^b-(y+z)^{b-\gamma}z^\gamma\in J$,

we get that

$$y^iz^{b+c-2-i}=y^by^{i-b}z^{b+c-2-i}=(y+z)^{b-\gamma}y^{i-b}z^{b+c-2-i+\gamma}$$
$$=\sum_{j=0}^{b-\gamma}\binom{b-\gamma}{j}y^{i-j-\gamma}z^{b+c-2-i+\gamma+j}\in J$$

by the induction hypothesis.

pISSN 1859-1388
eISSN 2615-9678

Remark 3.6. *Recently, the first author and Miró-Roig in [19] gave a general results on the WLP of* R/I *that recover the main result in Theorem 3.5. However, in this paper, we give another proof by using the properties of Hilbert function of* $A_\beta = R/I$ *and Proposition 2.5 to reduce to the more simply case where* $\beta = 1$.

Finally, we recover a result in [8, Theorem 3.7], with a shorter and easier proof.

Corollary 3.7. *Let* I *be as in Setting 3.1. If one of the* a, b *and* c *is equal to two, then* R/I *has the WLP.*

Proof. Recall that $b \le a+c-2$ and by symmetry of x and z, without loss of generality, we assume that $a \ge c$. Now, we consider the two cases.

Case 1. $b = 2$. Since $a \ge c$, therefore $a \ge 2b + c - 4$. By Theorem 3.5, R/I has the WLP.

Case 2. $c = 2$. Since $b \le a+c-2$, $a \ge b$, or equivalently $a \ge b+c-2$. Theorem 3.5 implies that if $a \ge 2b-2$, then R/I has the WLP for any $1 \le \beta \le b-1$ and if $b \le a \le 2b-3$, then R/I has the WLP for any $1 \le \beta \le a-b+1$. It remains to show that R/I has the WLP for any $a-b+2 \le \beta \le b-1$.

In this case, one has

$$I = (x^a, y^b - x^{b-1}z, z^2, x^{a-b+1}y^{b-\beta}, y^{b-\beta}z),$$

where $a-b+2 \le \beta \le b-1$. Note that the socle degree of R/I is $a+b-\beta-1$. If $\beta = a-b+2$, then the socle degree of R/I is odd, hence by applying Propositions 2.5 and 3.2, we can show that R/I has the WLP whenever $a-b+2 \le \beta \le b-1$ and $a+b-\beta$ is even. In this case, $k = \dfrac{a+b-\beta-2}{2}$. Set $L = x-y-z$. From Proposition 2.4, it is enough to show that the multiplication

$$\times L : [R/I]_k \to [R/I]_{k+1}$$

is surjective, or equivalently $[R/(I,L)]_{k+1} = 0$. We have that

$$R/(I,L) \cong K[y,z]/J$$

where

$$J = ((y+z)^a, y^b - (y+z)^{b-1}z, z^2, (y+z)^{a-b+1}y^{b-\beta}, y^{b-\beta}z).$$

We will prove that $[K[y,z]/J]_{k+1} = 0$, or equivalently $y^{k+1-i}z^i \in J$ for all $0 \le i \le k+1$.

Clearly, $y^{k+1-i}z^i \in J$ for all $2 \le i \le k+1$, as $z^2 \in J$. Since $k \ge b-\beta$,

$$y^k z = y^{b-\beta} z y^{k-b+\beta} \in J.$$

Finally, it is easy to see that $k \ge a-\beta$. Since $(y+z)^{a-b+1}y^{b-\beta} \in J$ and by the Newton binomial expansion formula

$$(y+z)^{a-b+1}y^{b-\beta} = \sum_{j=0}^{a-b+1} \binom{a-b+1}{j} y^{a-\beta+1-j}z^j,$$

we get

$$y^{a-\beta+1} = (y+z)^{a-b+1}y^{b-\beta} - \sum_{j=1}^{a-b+1} \binom{a-b+1}{j} y^{a-\beta+1-j}z^j \in J,$$

which shows that $y^{k+1} \in J$. The proof is completed.

References

1. Brenner H, Kaid A. Syzygy bundles on P2 and the weak Lefschetz property. Illinois Journal of Mathematics. 2007;51(4):1299-1308.
2. Harbourne B, Schenck H, Seceleanu A. Inverse systems, Gelfand-Tsetlin patterns and the weak Lefschetz property. Journal of the London Mathematical Society. 2011;84(3):712-730.
3. Mezzetti E, Miró-Roig RM, Ottaviani G. Laplace equations and the weak Lefschetz property. Canadian Journal of Mathematics. 2013;65(3):634-654.
4. Migliore JC, Miró-Roig RM, Nagel U. Monomial ideals, almost complete intersec tions and the weak Lefschetz property. Transactions of The American Mathematical Society. 2011;363(1):229-257.

5. Migliore JC, Miró-Roig RM, Nagel U. On the weak Lefschetz property for powers of linear forms. Algebra Number Theory. 2012;6(3):487-526.
6. Migliore JC, Nagel U. Survey article: a tour of the weak and strong Lefschetz prop erties. Journal of Commutative Algebra. 2013;5(3):329-358.
7. Miró-Roig RM, Tran QH. On the weak Lefschetz property for almost complete inter sections generated by uniform powers of general linear forms. Journal of Algebra. 2020;551:209-231.
8. Miró-Roig RM, Tran QH. The weak Lefschetz property for Artinian Gorenstein algebras of codimension three. Journal of Pure and Applied Algebra. 2020;224(7):106305.
9. Stanley RP. The number of faces of a simplicial convex polytope. Advances in Mathematics. 1980;35(3):236-238.
10. Harima T, Migliore JC, Nagel U, Watanabe J. The weak and strong Lefschetz properties for Artinian K-algebras. Journal of Algebra. 2003;262(1):99-126.
11. Cook D II. The Lefschetz properties of monomial complete intersections in positive characteristic. Journal of Algebra. 2012;369:42-58.
12. Cook D II, Nagel U. The weak Lefschetz property, monomial ideals, and lozenges. Illinois Journal of Mathematics. 2011;55(1):377-395.
13. Harima T. Characterization of Hilbert functions of Gorenstein Artin algebras with the weak Stanley property. Proceedings of The American Mathematical Society. 1995;123(12):3631-3638.
14. Migliore JC, Zanello F. The strength of the weak Lefschetz property. Illinois Journal of Mathematics. 2008;52(4):1417-1433.
15. Boij M, Migliore JC, Miró-Roig RM, Nagel U, Zanello F. On the weak Lefschetz property for Artinian Gorenstein algebras of codimension three. Journal of Algebra. 2014;403:48-68.
16. Bryant L. Goto numbers of a numerical semigroup ring and the Gorensteiness of associated graded rings. Communications in Algebra. 2010;38(6):2092-2128.
17. Guerrieri L. Lefschetz properties of Gorenstein graded algebras associated to the Apéry set of a numerical semigroup. Arkiv för Matematik. 2019;57(1):85-106.
18. Maeno T, Watanabe J. Lefschetz elements of Artinian Gorenstein algebras and Hessians of homogeneous polynomials. Illinois Journal of Mathematics. 2009;53(2):591-603.
19. Miró-Roig RM, Tran QH. The weak Lefschetz property of Gorenstein algebras of codimension three associated to the Apéry sets. Linear Algebra and its Applications. 2020;604:346-369.