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Coulomb divergence in (d,p) reactions based on Faddeev-Alt-Grassberger-Sandhas equation

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Abstract. The Faddeev-Alt-Grassberger-Sandhas equation is the most performance formalism to describe the (d,p) reactions since it can exactly and simultaneously treat the elastic, inelastic, transfer, and breakup reactions. In this brief report, we describe the progress and some main difficulties in treating the Coulomb interaction in the direct elastic scattering of the (d,p) reaction by using the Faddeev-Alt-Grassberger-Sandhas equation.

Keywords: (d,p) reactions, Faddeev-Alt-Grassberger-Sandhas equation, Coulomb problem

In the history of nuclear physics research, the (d,p) reaction is one of the most simple and powerful probes to study the nuclear structure. Huge experimental data have been obtained by using this reaction off the stable nuclei in the last century. Recently, this method continues to be the key to studying exotic nuclei in the framework of the inverse kinetic reaction. In principle, we can simplify the complicated $A+1$ problem to the 3-body problem by supposing the target as an inert core [1]. Because of this assumption, there are several approximations to solve this 3-body problem, such as Distorted Wave Born Approximation, Adiabatic Distorted Wave Approximation, Continuum Discretized Coupled-Channels method, and Faddeev/Alt, Grassberger, and Sandhas equations (FAGS). Nowadays, the FAGS is the standard method to benchmark every 3-body model since it is an exact model. However, because of the Coulomb divergence, the FAGS calculations are limited at the Ni isotopes [2-9]. The main underlying reason is that the Coulomb interaction increases in function of Z . We know very well that the Coulomb divergence appears in the triangular diagrams describing the $d+A$ and $p+(nA)$ elastic scattering with the four-ray vertex. Yet, the noncompact singularity appears because of the singularity of the off-shell $p+A$ Coulomb scattering amplitude, which coincides with the pole singularity of the two-body Green's function. We will show below the amplitude of the direct elastic scattering.

$$M_C(k'_\alpha, k_\alpha) = \int_0^\infty \frac{dp'_\alpha}{(2\pi)^3} \psi^{*(C)(-)}_{k'_\alpha}(p'_\alpha) \int_0^\infty \frac{dp_\alpha}{(2\pi)^3} \psi^{(C)(+)}_{k_\alpha}(p_\alpha) M_t(p'_\alpha, p_\alpha), \quad (1)$$

where

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$$M_t(p'_\alpha, p_\alpha) = 4\pi Z_\alpha Z_\beta e^2 \int_0^\infty \frac{dp_\gamma}{(2\pi)^3} \int_0^1 \mathrm{d}x\, x^{i\hat{\eta}^0_{\gamma\epsilon}} \frac{(1-x^2)}{[x\hat{\Delta}^2_\gamma + |b_\epsilon|(1-x)^2]^2}, \tag{2}$$

and

$$\hat{\Delta}^2_\gamma = (\mathrm{p}'_\alpha - \mathrm{p}_\alpha)^2, \tag{3}$$

and

$$b_\epsilon = \frac{\mu_\gamma}{2\hat{z}_\gamma}(\hat{z}_\gamma - \frac{k^2_\gamma}{2\mu_\gamma} + i\epsilon)(\hat{z}_\gamma - \frac{k'^2_\gamma}{2\mu_\gamma} + i\epsilon), \tag{4}$$

when $b = \lim_{\epsilon \to 0} b_\epsilon$,

where

$$\hat{z}_\gamma = z - \frac{p^2_\gamma}{2M_\gamma} \tag{5}$$

is the on-shell relative kinetic energy of β and α on the diagram, and where $z = E_{\alpha(\beta\gamma)} - \epsilon_{\beta\gamma}$ is the relative kinetic energy of α and $(\beta\gamma)$ when the on-shell $\epsilon_{\beta\gamma}$ is the binding energy of β and γ, $M_\alpha = M_{\alpha(\beta\gamma)} = \dfrac{m_\alpha m_{\beta\gamma}}{m_\alpha + m_\beta + m_\gamma}$.

If we neglect the re-scattering effects, the elastic scattering term is

$$M_C(k'_\alpha, k_\alpha) = \int_0^\infty \frac{dp'_\alpha}{(2\pi)^3} \psi^{*(C)(-)}_{k'_\alpha}(p'_\alpha) \int_0^\infty \frac{dp_\alpha}{(2\pi)^3} \psi^{(C)(+)}_{k_\alpha}(p_\alpha) M_t(p'_\alpha, p_\alpha), \tag{6}$$

where

$$M_t(p'_\alpha, p_\alpha) = 4\pi Z_d Z_A e^2 \int_0^1 \mathrm{d}x\, x^{i\eta_{dA}} \frac{1}{x\hat{\Delta}^2 - b(1-x)^2}, \tag{7}$$

and

$$\hat{\Delta}^2 = (\mathrm{p}'_\alpha - \mathrm{p}_\alpha)^2, \tag{8}$$

and

$$b = \frac{\mu_{dA}}{2\hat{z}_{dA}}(\hat{z}_{dA} - \frac{k^2_{dA}}{2\mu_{dA}})(\hat{z}_{dA} - \frac{k'^2_{dA}}{2\mu_{dA}}), \tag{9}$$

where

$$\hat{z}_{dA} = z = \frac{k^2_\alpha}{2M_\alpha} - \epsilon_{\beta\gamma}, \qquad \mu_{dA} = \frac{m_d m_A}{m_d + m_A}, \tag{10}$$

$$\frac{k^2_{dA}}{2\mu_{dA}} = \frac{p^2_\alpha}{2\mu_{dA}}, \qquad \frac{k'^2_{dA}}{2\mu_{dA}} = \frac{p'^2_\alpha}{2\mu_{dA}}, \tag{11}$$

Finally, we get

$$b = \frac{\mu_{dA}}{2\hat{z}_{dA}}(\frac{k^2_\alpha}{2M_\alpha} - \epsilon_d - \frac{p^2_\alpha}{2\mu_{dA}})(\frac{k'^2_\alpha}{2M_\alpha} - \epsilon_d - \frac{p'^2_\alpha}{2\mu_{dA}}), \tag{12}$$

$$M_C(k'_\alpha, k_\alpha) = \int_0^\infty \frac{dp'_\alpha}{(2\pi)^3} \psi^{*(C)(-)}_{k'_\alpha}(p'_\alpha) \int_0^\infty \frac{dp_\alpha}{(2\pi)^3} .\psi^{(C)(+)}_{k_\alpha}(p_\alpha) M_t(p'_\alpha, p_\alpha) \tag{13}$$

After some non-trivial analytical transformations in using the Nordsieck integral, we get

$$M_C(k'_\alpha, k_\alpha) = 4\pi Z_d Z_A e^2 \int_0^\infty \frac{dp'_\alpha}{(2\pi)^3} M_L(k_\alpha, p'_\alpha) \psi^{*(C)(-)}_{k_\alpha}(p'_\alpha), \tag{14}$$

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where $$M_L(k_\alpha, p'_\alpha) = \int_0^1 dx \frac{x^{i\eta_{dA}}}{C_3^2 x} e^{-\frac{\pi\hat{\eta}_{dA}^0}{2}} \Gamma(1+i\eta_{dA}) \frac{[\frac{p_\alpha'^2}{C_3^4} - (k_\alpha + i\kappa)^2]^{i\eta_{dA}}}{[(\frac{p'_\alpha}{C_3^2} - k_\alpha)^2 + \kappa^2]^{1+i\eta_{dA}}}, \tag{15}$$

and

$$C_0 = (\hat{z}_{dA} - \frac{p_\alpha'^2}{2\mu_{dA}}) \frac{\mu_{dA}}{2\hat{z}_{dA}}, \tag{16}$$

$$C_1 = -C_0 \frac{(1-x)^2}{x} (\hat{z}_{dA}), \tag{17}$$

$$C_2 = \frac{C_0}{2\mu_{dA}} \frac{(1-x)^2}{x}, \tag{18}$$

$$C_3^2 = 1 + C_2, \tag{19}$$

$$\kappa^2 = \frac{(1-\frac{1}{C_3^2}) p_\alpha'^2 + C_1}{C_3^2}. \tag{20}$$

To calculate the integral of Eq. (17), we use one more approximation when supposing $p'_\alpha = k_\alpha$. This approximation works near the forward singularity $p'_\alpha = k'_\alpha$. Using the Cauchy integral formula and Nordsieck integral, we get

$$M_C(k'_\alpha, k_\alpha) = 4\pi Z_d Z_A e^2 e^{-\pi\eta_{dA}} \Gamma(1+i\eta_{dA}) \frac{e^{i\pi\lambda} - e^{-i\pi\lambda}}{2\pi i} \int_0^1 dx \frac{x^{i\eta_{dA}}}{x} C_3^{(2+4i\eta_{dA})} [k_\alpha^2 - (k_\alpha + i\kappa)^2]^{i\eta_{dA}} \int_0^\infty dt \frac{1}{t^\lambda}$$

$$\times \Gamma(1+i\eta_{dA}) \frac{[\tilde{k}_\alpha^2 - (k'_\alpha + i\chi_1)^2]^{i\eta_{dA}}}{[(\tilde{k}_\alpha - k'_\alpha)^2 + \chi_1^2]^{1+i\eta_{dA}}} \tag{21}$$

$$= 4\pi Z_d Z_A e^2 e^{-\pi\eta_{dA}} \Gamma^2(1+i\eta_{dA}) \frac{\cos(\pi\lambda)}{\pi} \int_0^1 \frac{x^{i\eta_{dA}}}{x} C_3^{(2+4i\eta_{dA})} [k_\alpha^2 - (k_\alpha + i\kappa)^2]^{i\eta_{dA}}$$

$$\times \int_0^\infty dt \frac{1}{t^\lambda} \frac{[C_3^4 k_\alpha^2 - (k'_\alpha + i\chi_1)^2]^{i\eta_{dA}}}{[(C_3^2 k_\alpha - k'_\alpha)^2 + \chi_1^2]^{1+i\eta_{dA}}} \tag{22}$$

$$= 4\pi Z_d Z_A e^2 e^{-\pi\eta_{dA}} \Gamma^2(1+i\eta_{dA}) \frac{\cos(\pi\lambda)}{\pi} \int_0^1 dx \frac{x^{i\eta_{dA}}}{x} C_3^{(2+4i\eta_{dA})} [k_\alpha^2 - (k_\alpha + i\kappa)^2]^{i\eta_{dA}}$$

$$\times \int_0^\infty \frac{d\rho}{C_4^{\lambda-1} \rho^\lambda} \frac{C_5^{i\eta_{dA}}}{C_4^{1+i\eta_{dA}} (1+\rho)^{1+i\eta_{dA}}} \tag{23}$$

$$= 4\pi Z_d Z_A e^2 e^{-\pi\eta_{dA}} \Gamma^2(\lambda) \frac{\cos(\pi\lambda)}{\pi} \int_0^1 dx \frac{x^{i\eta_{dA}}}{x} C_3^{4\lambda-2} [k_\alpha^2 - (k_\alpha + i\kappa)^2]^{i\eta_{dA}}$$

$$\times \int_0^\infty \frac{d\rho}{C_4^{i\eta_{dA}} \rho^\lambda} \frac{C_5^{i\eta_{dA}}}{C_4^\lambda (1+\rho)^\lambda} \tag{24}$$

$$= 4\pi Z_d Z_A e^2 e^{-\pi\eta_{dA}} \Gamma^2(\lambda) \frac{\cos(\pi\lambda)}{\pi} \int_0^1 dx \frac{x^{i\eta_{dA}}}{x} C_3^{4\lambda-2} (-2ik_\alpha\kappa + \kappa^2)^{i\eta_{dA}} \int_0^\infty \frac{C_5^{\lambda-1} d\rho}{C_4^{2\lambda-1} \rho^\lambda (1+\rho)^\lambda} \tag{25}$$

where

$$\lambda = 1 + i\eta_{dA} \tag{26}$$

$$\tilde{k}_\alpha = C_3^2 k_\alpha \tag{27}$$

$$\chi^2 = C_3^4 \kappa^2 \tag{28}$$

$$\chi_1^2 = \chi^2 + t \tag{29}$$

$$C_4 = (C_3^2 k_\alpha - k'_\alpha)^2 + \chi^2 \tag{30}$$

$$C_5 = C_3^4 k_\alpha^2 - (k'_\alpha + i\chi)^2 \tag{31}$$

$$t = C_4 \rho \tag{32}$$

$$\eta_{dA} = \frac{\mu_{dA} Z_d Z_A e^2}{k_\alpha} \quad (33)$$

We expand $M_C(k'_\alpha, k_\alpha)$ in a series in partial waves

$$M_C(k'_\alpha, k_\alpha) = \sum_{l=0}^{\infty}(2l+1)P_l(Z)M_{Cl}(k'_\alpha, k_\alpha), \qquad Z = \cos(\theta) = \hat{k}'_\alpha \hat{k}_\alpha \quad (34)$$

and the partial-wave amplitudes M_{Cl} are determined by

$$M_{Cl}(k'_\alpha, k_\alpha) = \frac{1}{2}\int_{-1}^{1} dZ P_l(Z) M_C(k'_\alpha, k_\alpha)$$

$$= 4\pi Z_d Z_A e^2 e^{-\pi\eta_{dA}} \Gamma^2(\lambda) \frac{\sin(\pi\lambda)}{\pi} \frac{1}{2}\int_{-1}^{1} dZ C_4^{-2\lambda+1} P_l(Z) M_C(k'_\alpha, k_\alpha) \int_0^1 dx \frac{x^{i\eta_{dA}}}{x} C_3^{4\lambda-2}(-2ik_\alpha\kappa + \kappa^2)^{i\eta_{dA}}$$

$$\times \int_0^\infty \frac{C_5^{\lambda-1} d\rho}{\rho^\lambda (1+\rho)^\lambda} \quad (35)$$

$$= 4\pi Z_d Z_A e^2 e^{-\pi\eta_{dA}} \Gamma^2(\lambda) \frac{\sin(\pi\lambda)}{\pi} \frac{1}{2}\int_{-1}^{1} dZ (C_6 - Z)^{-2\lambda+1} (2C_3^2 k_\alpha k'_\alpha)^{-2\lambda+1} P_l(Z) M_C(k'_\alpha, k_\alpha)$$

$$\times \int_0^1 dx \frac{x^{i\eta_{dA}}}{x} C_3^{4\lambda-2}(-2ik_\alpha\kappa + \kappa^2)^{i\eta_{dA}} \int_0^\infty \frac{C_5^{\lambda-1} d\rho}{\rho^\lambda (1+\rho)^\lambda} \quad (36)$$

$$= 8\pi Z_d Z_A e^2 e^{-\pi\eta_{dA}} \Gamma^2(\lambda) \frac{\sin(\pi\lambda)}{\pi} 4^{-\lambda} (k_\alpha k'_\alpha)^{-2\lambda+1} \frac{1}{2}\int_{-1}^{1} dZ (C_6 - Z)^{-2\lambda+1} P_l(Z) M_C(k'_\alpha, k_\alpha)$$

$$\times \int_0^1 dx \frac{x^{i\eta_{dA}}}{x} (-2ik_\alpha\kappa + \kappa^2)^{i\eta_{dA}} \int_0^\infty \frac{C_5^{\lambda-1} d\rho}{\rho^\lambda (1+\rho)^\lambda} \quad (37)$$

$$= -8\pi Z_d Z_A e^2 e^{-\pi\eta_{dA}} \Gamma^2(\lambda) \frac{\sin(\pi\lambda)}{\pi} 4^{-\lambda} (k_\alpha k'_\alpha)^{-2\lambda+1} \frac{e^{-i\pi(2\lambda-1)}}{\Gamma(2\lambda-1)}$$

$$\times \int_0^1 dx \frac{x^{i\eta_{dA}}}{x} [-2ik_\alpha\kappa(x) + \kappa^2(x)]^{i\eta_{dA}} [C_6^2(x) - 1]^{1-\lambda} Q_l^{2\lambda-2}[C_6(x)] \int_0^\infty \frac{C_5^{\lambda-1} d\rho}{\rho^\lambda (1+\rho)^\lambda} \quad (38)$$

$$= -8\pi Z_d Z_A e^2 e^{-\pi\eta_{dA}} \Gamma^2(\lambda) \frac{\sin(\pi\lambda)}{\pi} 4^{-\lambda} (k_\alpha k'_\alpha)^{-2\lambda+1} \frac{e^{-i\pi(2\lambda-1)}}{\Gamma(2\lambda-1)}$$

$$\times \int_0^1 dx \frac{x^{i\eta_{dA}}}{x} [-2ik_\alpha\kappa(x) + \kappa^2(x)]^{i\eta_{dA}} C_5^{\lambda-1}(x) [C_6^2(x) - 1]^{1-\lambda} Q_l^{2\lambda-2}[C_6(x)] \int_0^\infty \frac{d\rho}{\rho^\lambda (1+\rho)^\lambda} \quad (39)$$

$$= -8\pi Z_d Z_A e^2 e^{-\pi\eta_{dA}} \Gamma^2(\lambda) \frac{\sin(\pi\lambda)}{\pi} 4^{-\lambda} (k_\alpha k'_\alpha)^{-2\lambda+1} \frac{e^{-i\pi(2\lambda-1)}}{\Gamma(2\lambda-1)}$$

$$\times \int_0^1 dx \frac{x^{i\eta_{dA}}}{x} [-2ik_\alpha\kappa(x) + \kappa^2(x)]^{i\eta_{dA}} C_5^{\lambda-1}(x) [C_6^2(x) - 1]^{1-\lambda} Q_l^{2\lambda-2}[C_6(x)] \frac{\Gamma(-i\eta_{dA})\Gamma(1+2i\eta_{dA})}{\Gamma(1+i\eta_{dA})} \quad (40)$$

$$= -8\pi Z_d Z_A e^2 e^{-\pi\eta_{dA}} \Gamma(\lambda)\Gamma(-i\eta_{dA}) \frac{\sin(\pi\lambda)}{\pi} 4^{-\lambda} (k_\alpha k'_\alpha)^{-2\lambda+1} e^{-i\pi(2\lambda-1)}$$

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$$\times\int_0^1 dx \frac{x^{i\eta_{dA}}}{x}[-2ik_\alpha \kappa(x)+\kappa^2(x)]^{i\eta_{dA}} C_5^{\lambda-1}(x)[C_6^2(x)-1]^{1-\lambda} Q_l^{2\lambda-2}[C_6(x)] \quad (41)$$

$$= -8\pi Z_d Z_A e^2 e^{-\pi\eta_{dA}} \Gamma(\lambda)\Gamma(-i\eta_{dA}) \frac{\sin(\pi\lambda)}{\pi} 4^{-\lambda}(k_\alpha k'_\alpha)^{-2\lambda+1} e^{-i\pi(2\lambda-1)}$$

$$\times\int_0^1 dx \frac{x^{i\eta_{dA}}}{x}[-2ik_\alpha \kappa(x)+\kappa^2(x)]^{i\eta_{dA}} C_5^{i\eta_{dA}}(x)[\xi^2-1]^{-i\eta_{dA}} Q_l^{2i\eta_{dA}}(\xi) \quad (42)$$

where

$$C_4 = C_3^4 k_\alpha^2 + k_\alpha'^2 - 2C_3^2 k_\alpha k'_\alpha Z + \chi^2 \quad (43)$$

$$C_5 = C_3^4 k_\alpha^2 - (k'_\alpha + i\chi)^2 \quad (44)$$

$$C_6 = \frac{C_3^4 k_\alpha^2 + k_\alpha'^2 + \chi^2}{2C_3^2 k_\alpha k'_\alpha} = \xi \quad (45)$$

and $Q_l^{2i\eta_{dA}}(\xi)$ is the Associated Legendre Function of the second kind, which could be expressed in terms of the hypergeometric function ${}_2F_1(a,b;c;z)$.

To test the formalism, we perform the calculation for the deuteron elastic scattering off Pb ^{208} at 9 MeV. Unfortunately, the obtained results (Fig. 1) show that the Coulomb divergence is still very strong even though we already subtracted the first-order Born term as in Ref. [10]. This is still a very hard barrier to overcome. In the near future, we propose to use the screening and renormalization method for the higher order of the Coulomb term after the subtraction of the first order.

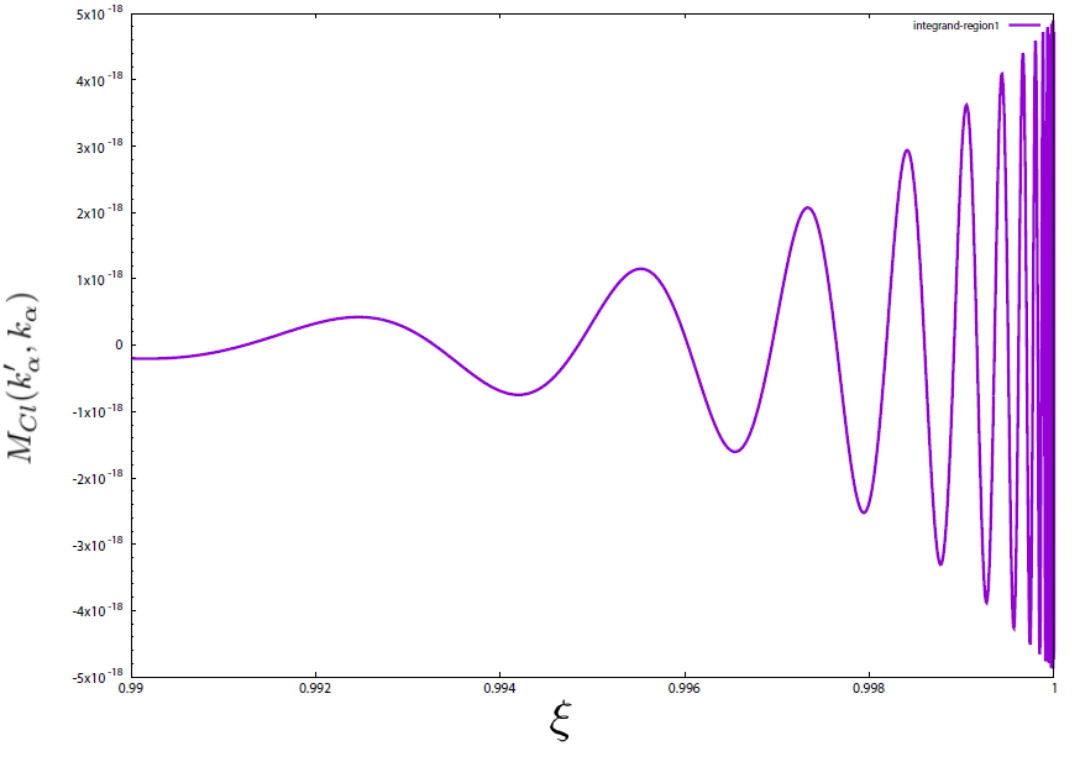

Fig. 1. (Color line) The amplitude of elastic scattering of deuteron off Pb ^{208} at 9 MeV

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