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Some remarks on Pixley-Roy hyperspaces

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Abstract. In [1], H.T.O. Trieu, L.Q. Tuyen and O.V. Tuyen proved that a space X is countable if and only if $\mathtt{PR}[X]$ is strongly star-Lindelöf. In this paper, we study the extension of cellular-compact, cellular-Lindelöf, strongly star-Hurewicz, strongly star-Rothberger, strongly star-Menger spaces on hyperspaces with the Pixley-Roy topology. For a space X and $n \in \mathbb{N}$, we prove that

(a) If $\mathtt{PR}[X]$ or $\mathtt{PR}_n[X]$ is cellular-compact, then X is cellular-compact. However, there exists a compact space X such that $|X| = \omega$, but $\mathtt{PR}_n[X]$ for all $n \in \mathbb{N}$ and $\mathtt{PR}[X]$ are not cellular-compact spaces.

(b) If $\mathtt{PR}[X]$ or $\mathtt{PR}_n[X]$ is cellular-Lindelöf, then X is cellular-Lindelöf.

(c) X is countable if and only if $\mathtt{PR}[X]$ is a strongly star-Hurewicz space, if and only if $\mathtt{PR}[X]$ is a strongly star-Rothberger space if and only if $\mathtt{PR}[X]$ is a strongly star-Menger space.

Keywords: Hyperspace, Pixley-Roy topology, cellular-compact, cellular-Lindelöf, strongly star-Hurewicz, strongly star-Rothberger, strongly star-Menger

1 Introduction and preliminaries

The topological and generalized metric properties on Pixley–Roy hyperspaces have been studied by many authors from different points of view. Specifically, they studied the relation between a space X satisfying such a property and its Pixley-Roy hyperspaces $\mathtt{PR}[X]$ and $\mathtt{PR}_n[X]$ satisfying the same property [1, 2, 3, 4, 5, 6, 7]. Li Z. demonstrated important results concerning selection principles, such as quasi-Hurewicz, quasi-Rothberger, quasi-Menger (see [2]), and Hurewicz separability (see [3]), within the framework of the Pixley-Roy hyperspace $\mathtt{PR}[X]$. In [1], H.T.O. Trieu, L.Q. Tuyen and O.V. Tuyen proved that a space X is countable if and only if $\mathtt{PR}[X]$ is strongly star-Lindelöf.

Recently, the new generalized metric properties, such as cellular-compact, cellular-Lindelöf, strongly star-Hurewicz, strongly star-Rothberger, and strongly star-Menger spaces, are introduced in [8, 9, 10]. These properties generalize classical notions and offer deeper insights into hyperspace topology. Studying these in the Pixley-Roy setting is motivated by their applications to function spaces, selection principles, and covering properties, as well as their ability to illuminate how such properties transfer between a space X and its hyperspaces. In this paper, we study them on hyperspaces with the Pixley-Roy topology. For a space X and $n \in \mathbb{N}$, we prove that

(a) If $\mathtt{PR}[X]$ or $\mathtt{PR}_n[X]$ is cellular-compact, then X is cellular-compact. However, there exists a compact space X such that $|X| = \omega$, but $\mathtt{PR}_n[X]$ for all $n \in \mathbb{N}$ and $\mathtt{PR}[X]$ are not cellular-compact spaces.

(b) If $\mathtt{PR}[X]$ or $\mathtt{PR}_n[X]$ is cellular-Lindelöf, then X is cellular-Lindelöf.

And, the following statements are equivalent:

(a) X is countable.

(b) $\mathtt{PR}[X]$ is a strongly star-Hurewicz space.

(c) $\mathtt{PR}[X]$ is a strongly star-Rothberger space.

(d) $\mathtt{PR}[X]$ is a strongly star-Menger space.

Throughout this paper, all spaces are assumed to be Hausdorff and $\mathbb{N}$ denotes the set of all positive integers with the first infinite ordinal denoted by ω. Moreover, if $\mathcal{P}$ is a family of subsets of a space X and $A \subset X$, denote

$$\mathtt{St}(A, \mathcal{P}) = \bigcup\{P \in \mathcal{P} : P \cap A \neq \emptyset\};$$
$$\bigcup \mathcal{P} = \bigcup\{P : P \in \mathcal{P}\}.$$

The set $\mathtt{PR}[X]$ is the set of all non-empty finite subsets of a space X. For each $F \in \mathtt{PR}[X]$ and $A \subset X$, denote

$$[F, A] = \{H \in \mathtt{PR}[X] : F \subset H \subset A\}.$$

The *Pixley-Roy hyperspace* $\mathtt{PR}[X]$ over a space X is defined by C. Pixley and P. Roy in [11], with the topology generated by the sets of the form $[F, V]$, where $F \in \mathtt{PR}[X]$ and V is an open subset in X containing F. For any space X, $\mathtt{PR}[X]$ is zero-dimensional, completely regular and hereditarily metacompact (see [12]).

For each $n \in \mathbb{N}$, let

$$\mathtt{PR}_n[X] = \{F \in \mathtt{PR}[X] : |F| \leq n\}.$$

Then,

$$\mathtt{PR}[X] = \bigcup_{n=1}^{\infty} \mathtt{PR}_n[X]$$

and $\mathtt{PR}_n[X] \subset \mathtt{PR}_{n+1}[X]$ for each $n \in \mathbb{N}$.

Remark 1.1. Let X be a space and $n \in \mathbb{N}$.

(a) $\mathtt{PR}_n[X]$ is a closed subspace of $\mathtt{PR}[X]$ and in particular, $\mathtt{PR}_1[X]$ is a closed discrete subspace of $\mathtt{PR}[X]$ [13].

(b) Every $\mathtt{PR}_m[X]$ is a closed subspace of $\mathtt{PR}_n[X]$ for each $m, n \in \mathbb{N}$, $m < n$ [5].

Definition 1.2. Let X be a space.

(a) X is *Lindelöf* [14], if for any open cover of X, there exists a countable subcover.

(b) X is *cellular-Lindelöf* [8], if for any disjoint family $\mathcal{U}$ of non-empty open sets, there is a Lindelöf subspace L such that $L \cap U \neq \emptyset$ for each $U \in \mathcal{U}$.

(c) X is *cellular-compact* [9], if for any disjoint family $\mathcal{U}$ of non-empty open sets, there is a compact subspace K such that $K \cap U \neq \emptyset$ for each $U \in \mathcal{U}$.

(d) X is *strongly star-Hurewicz* [10], if for each sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, there exists a sequence $\{F_n : n \in \omega\}$ of finite subsets of X such that for each $x \in X$, $x \in \mathtt{St}(F_n, \mathcal{U}_n)$ for all but finitely many n.

(e) X is *strongly star-Menger* [10], if for each sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, there exists a sequence $\{F_n : n \in \omega\}$ of finite subsets of X such that

$$\{\mathtt{St}(F_n, \mathcal{U}_n) : n \in \omega\}$$

is an open cover of X.

(f) X is *strongly star-Rothberger* [10], if for each sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X, there exists a sequence

$$\{x_n : n \in \omega\}$$

of elements of X such that

$$\{\mathtt{St}(x_n, \mathcal{U}_n) : n \in \omega\}$$

is an open cover of X.

Remark 1.3. (a) Compact $\Rightarrow$ cellular-compact $\Rightarrow$ cellular-Lindelöf.

(b) Compact ⇒ Lindelöf ⇒ cellular-Lindelöf.

(c) Strongly star-Hurewicz ⇒ strongly star-Menger

(d) Strongly star-Rothberger ⇒ strongly star-Menger

2 Main result

Lemma 2.1 ([5], Lemma 2)**.** *Let X be a space. If $\mathcal{C}$ is a compact subset of $\mathtt{PR}[X]$, then $\bigcup\mathcal{C}$ is a compact subset of X.*

Theorem 2.2. *Let X be a space and $n \in \mathbb{N}$. If $\mathtt{PR}[X]$ or $\mathtt{PR}_n[X]$ is cellular-compact, then X is cellular-compact.*

Proof. Let $\mathtt{PR}[X]$ be cellular-compact and $\mathcal{U}$ be a disjoint family of non-empty open sets in X. For each $U \in \mathcal{U}$, take $x \in U$ and put

$$\mathfrak{U} = \Big\{[\{x\}, U] : U \in \mathcal{U}\Big\}.$$

Then, $\mathfrak{U}$ is a disjoint family of non-empty open sets in $\mathtt{PR}[X]$. Since $\mathtt{PR}[X]$ is cellular-compact, there exists a compact subspace $\mathcal{K}$ of $\mathtt{PR}[X]$ such that

$$\mathcal{K} \cap [\{x\}, U] \neq \emptyset$$

for each $U \in \mathcal{U}$. Thus, for each $U \in \mathcal{U}$, there exists $F \in \mathcal{K}$ such that

$$\{x\} \subset F \subset U.$$

This implies that

$$\left(\bigcup\mathcal{K}\right) \cap U \neq \emptyset$$

for each $U \in \mathcal{U}$. Moreover, $\bigcup\mathcal{K}$ is compact in X by Lemma 2.1. Therefore, X is cellular-compact.

Similar to the above proof, we claim that for each $n \in \mathbb{N}$, if $\mathtt{PR}_n[X]$ is cellular-compact, then X is cellular-compact. □

Example 2.3. There exists a compact space X such that $|X| = \omega$, but $\mathtt{PR}_n[X]$ for all $n \in \mathbb{N}$ and $\mathtt{PR}[X]$ are not cellular-compact spaces.

Proof. Assume that

$$X = \{x_0\} \cup \{x_k : k \in \mathbb{N}\},$$

where every x_k and x_0 are different from each other. The set X endowed with the following topology: each x_k is isolated; a basic neighborhood of x_0 has the form

$$\{x_0\} \cup \{x_k : k \geq m\}$$

for some $m \in \mathbb{N}$.

(1) It is obvious that X is compact.

(2) $\mathtt{PR}[X]$ is not cellular-compact. In fact, let

$$\mathfrak{U} = \Big\{[\{x_k\}, \{x_k\}] : k \in \mathbb{N}\Big\}.$$

Then, $\mathfrak{U}$ is a disjoint family of non-empty open sets in $\mathtt{PR}[X]$. We prove that for each $\mathcal{K}$ compact in $\mathtt{PR}[X]$, there exists $k \in \mathbb{N}$ such that

$$\mathcal{K} \cap [\{x_k\}, \{x_k\}] = \emptyset.$$

Otherwise, assume that there exists $\mathcal{K}$ compact in $\mathtt{PR}[X]$ such that for each $k \in \mathbb{N}$, we have

$$\mathcal{K} \cap [\{x_k\}, \{x_k\}] \neq \emptyset.$$

Then, $\{x_k\} \in \mathcal{K}$. We put

$$\mathcal{K}_1 = \mathcal{K} \setminus \{\{x_k\} : k \in \mathbb{N}\}.$$

Thus,

$$\mathfrak{V} = \mathfrak{U} \cup \{[F, X] : F \in \mathcal{K}_1\}$$

is an open cover of $\mathcal{K}$ in $\mathtt{PR}[X]$. Since for each $F \in \mathcal{K}_1$, or $F = \{x_0\}$ or $|F| \geq 2$, we claim that

$$\{x_k\} \notin [F, X]$$

for each $k \in \mathbb{N}$ and for each $F \in \mathcal{K}_1$. This implies that $\mathfrak{V}$ does not have any finite subcover of $\mathcal{K}$, which is a contradiction.

(3) $\mathtt{PR}_n[X]$ is not cellular-compact for all $n \in \mathbb{N}$. In fact, let

$$\mathfrak{U} = \Big\{[\{x_k\}, \{x_k\}] : k \in \mathbb{N}\Big\}.$$

Then, $\mathfrak{U}$ is a disjoint family of non-empty open sets in $\mathtt{PR}_n[X]$. We prove that for each $\mathcal{K}$ compact in $\mathtt{PR}_n[X]$, there exists $k \in \mathbb{N}$ such that

$$\mathcal{K} \cap [\{x_k\}, \{x_k\}] = \emptyset.$$

Otherwise, assume that there exists $\mathcal{K}$ compact in $\mathtt{PR}_n[X]$ such that for each $k \in \mathbb{N}$, we have

$$\mathcal{K} \cap [\{x_k\}, \{x_k\}] \neq \emptyset.$$

Then, $\{x_k\} \in \mathcal{K}$. We put

$$\mathcal{K}_1 = \mathcal{K} \setminus \{\{x_k\} : k \in \mathbb{N}\}.$$

Thus,

$$\mathfrak{V} = \mathfrak{U} \cup \Big\{[F, X] \cap \mathtt{PR}_n[X] : F \in \mathcal{K}_1\Big\}$$

is an open cover of $\mathcal{K}$ in $\mathtt{PR}_n[X]$. Since for each $F \in \mathcal{K}_1$, or $F = \{x_0\}$ or

$$2 \leq |F| \leq n,$$

we claim that

$$\{x_k\} \notin [F, X] \cap \mathtt{PR}_n[X]$$

for each $k \in \mathbb{N}$ and for each $F \in \mathcal{K}_1$. This implies that $\mathfrak{V}$ does not have any finite subcover of $\mathcal{K}$, which is a contradiction. □

Lemma 2.4. *Let X be a space. If $\mathcal{L}$ is a Lindelöf subset of $\mathtt{PR}[X]$, then $\bigcup \mathcal{L}$ is a Lindelöf subset of X.*

Proof. Let $\mathcal{U}$ be an open cover of $\bigcup \mathcal{L}$ in X. Then, for each $F \in \mathcal{L}$, we have

$$F \subset \bigcup \mathcal{L} \subset \bigcup \mathcal{U}.$$

Thus, for each $x \in F$, there exists $U_x \in \mathcal{U}$ such that $x \in U_x$. Since $F \in [F, \bigcup_{x \in F} U_x]$ for each $F \in \mathcal{L}$, we claim that

$$\mathfrak{U} = \Big\{[F, \textstyle\bigcup_{x \in F} U_x] : F \in \mathcal{L}\Big\}$$

is an open cover of $\mathcal{L}$ in $\mathtt{PR}[X]$. On the other hand, since $\mathcal{L}$ is Lindelöf, there exists a countable subfamily $\mathfrak{V}$ of $\mathfrak{U}$ such that $\mathcal{L} \subset \bigcup \mathfrak{V}$. This implies that there exists $\{F_i : i \in \mathbb{N}\} \subset \mathcal{L}$ such that

$$\mathfrak{V} = \Big\{[F_i, \textstyle\bigcup_{x \in F_i} U_x] : i \in \mathbb{N}\Big\}.$$

Put

$$\mathcal{V} = \Big\{U_x : x \in F_i, i \in \mathbb{N}\Big\}.$$

Because each set F_i is finite, $\mathcal{V}$ is a countable subfamily of $\mathcal{U}$. Hence, we only need to prove that

$$\bigcup \mathcal{L} \subset \bigcup \mathcal{V}.$$

Take any point $z \in \bigcup \mathcal{L}$, then $z \in A$ for some $A \in \mathcal{L}$. This implies that there exists $i \in \mathbb{N}$ such that

$$A \in [F_i, \textstyle\bigcup_{x \in F_i} U_x],$$

hence $A \subset \bigcup_{x \in F_i} U_x$. Therefore,

$$z \in U_x \subset \bigcup \mathcal{V}$$

for some $x \in F_i$. □

Theorem 2.5. *Let X be a space and $n \in \mathbb{N}$. If $\mathtt{PR}[X]$ or $\mathtt{PR}_n[X]$ is cellular-Lindelöf, then X is cellular-Lindelöf.*

Proof. Let $\mathtt{PR}[X]$ be cellular-Lindelöf and $\mathcal{U}$ be a disjoint family of non-empty open sets in X. For each $U \in \mathcal{U}$, take $x \in U$ and put

$$\mathfrak{U} = \Big\{[\{x\}, U] : U \in \mathcal{U}\Big\}.$$

Then, $\mathfrak{U}$ is a disjoint family of non-empty open sets in $\mathtt{PR}[X]$. Since $\mathtt{PR}[X]$ is cellular-Lindelöf, there exists a Lindelöf subspace $\mathcal{L}$ of $\mathtt{PR}[X]$ such that

$$\mathcal{L} \cap [\{x\}, U] \neq \emptyset$$

for each $U \in \mathcal{U}$. Thus, for each $U \in \mathcal{U}$, there exists $F \in \mathcal{L}$ such that

$$\{x\} \subset F \subset U.$$

This implies that

$$(\bigcup \mathcal{L}) \cap U \neq \emptyset$$

for each $U \in \mathcal{U}$. Moreover, $\bigcup \mathcal{L}$ is Lindelöf in X by Lemma 2.4. Therefore, X is cellular-Lindelöf.

Similar to the above proof, we claim that for each $n \in \mathbb{N}$, if $\mathtt{PR}_n[X]$ is cellular-Lindelöf, then X is cellular-Lindelöf. □

For a space X, it follows from Remark 1.1(1) and[15, Theorem 3.1] that $\mathtt{PR}_1[X]$ is not cellular-Lindelöf if $\mathtt{PR}_1[X]$ is uncountable. On the other hand, X is countable if and only if $\mathtt{PR}_1[X]$ is countable. Therefore, we obtain the following corollary.

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Corollary 2.6. *Let X be a space. Then, X is countable if and only if $\mathtt{PR}_1[X]$ is cellular-Lindelöf.*

Question 2.1. *If X is cellular-Lindelöf, then are $\mathtt{PR}[X]$ and $\mathtt{PR}_n[X]$ for some $n \in \mathbb{N}$ cellular-Lindelöf?*

Question 2.2. *Is Theorem 2.5 still true if the hyperspace with the Pixley-Roy topology is replaced by the hyperspace equipped with either the Vietoris topology or the Fell topology?*

Theorem 2.7. *Let X be a space. Then, the following statements are equivalent:*

(a) X is countable.

(b) $\mathtt{PR}[X]$ is a strongly star-Hurewicz space.

(c) $\mathtt{PR}[X]$ is a strongly star-Rothberger space.

(d) $\mathtt{PR}[X]$ is a strongly star-Menger space.

Proof. (a) $\Rightarrow$ (b). Assume that X is countable and

$$\{\mathfrak{U}_n : n \in \omega\}$$

is a sequence of open covers of $\mathtt{PR}[X]$. Because X is countable, $\mathtt{PR}[X]$ is countable. We put

$$\mathtt{PR}[X] = \{F_n : n \in \omega\},$$

and for every $n \in \omega$, we put

$$\mathcal{A}_n = \{F_0, F_1, \ldots, F_n\}.$$

Then,

$$\{\mathcal{A}_n : n \in \omega\}$$

is a sequence of finite subsets of $\mathtt{PR}[X]$. Furthermore, for each $F \in \mathtt{PR}[X]$, we have

$$|\{n \in \omega : F \notin \mathtt{St}(\mathcal{A}_n, \mathfrak{U}_n)\}| < \omega.$$

Indeed, let $F \in \mathtt{PR}[X]$, then there exists $m \in \omega$ such that

$$F = F_m.$$

For every $n \geq m$, because $F_m \in \mathcal{A}_n$, it implies that

$$F = F_m \in \mathtt{St}(\mathcal{A}_n, \mathfrak{U}_n).$$

Thus,

$$|\{n \in \omega : H \notin \mathtt{St}(\mathcal{A}_n, \mathfrak{U}_n)\}| \leq m < \omega.$$

Therefore, $\mathtt{PR}[X]$ is a strongly star-Hurewicz space.

(b) $\Rightarrow$ (d). Let $\mathtt{PR}[X]$ be a strongly star-Hurewicz space and

$$\{\mathfrak{U}_n : n \in \omega\}$$

is a sequence of open covers of $\mathtt{PR}[X]$. Since $\mathtt{PR}[X]$ is a strongly star-Hurewicz space, there exists a sequence

$$\{\mathcal{A}_n : n \in \omega\}$$

of finite subsets of $\mathtt{PR}[X]$ such that for each $F \in \mathtt{PR}[X]$, we have

$$|\{n \in \omega : F \notin \mathtt{St}(\mathcal{A}_n, \mathfrak{U}_n)\}| < \omega.$$

Then,

$$\{\mathtt{St}(\mathcal{A}_n, \mathfrak{U}_n) : n \in \omega\}$$

is an open cover of $\mathtt{PR}[X]$. Therefore, $\mathtt{PR}[X]$ is a strongly star-Menger space.

(a) $\Rightarrow$ (c). Let X be countable and

$$\{\mathfrak{U}_n : n \in \omega\}$$

be a sequence of open covers of $\mathtt{PR}[X]$. Since X is countable, $\mathtt{PR}[X]$ is countable. Denote

$$\mathtt{PR}[X] = \{F_n : n \in \omega\}.$$

Then,

$$\{\mathtt{St}(F_n, \mathfrak{U}_n) : n \in \omega\}$$

is an open cover of $\mathtt{PR}[X]$. In fact, let $E \in \mathtt{PR}[X]$, then there exists $n_E \in \omega$ such that

$$E = F_{n_E}.$$

Since $\mathfrak{U}_{n_E}$ is an open cover of $\mathtt{PR}[X]$, there exists $\mathcal{U}_{n_E}$ such that $E \in \mathcal{U}_{n_E}$. Thus,

$$E \in \mathtt{St}(F_{n_E}, \mathfrak{U}_{n_E}).$$

Hence, $\mathtt{PR}[X]$ is a strongly star-Rothberger space.

(c) $\Rightarrow$ (d). Let $\mathtt{PR}[X]$ be a strongly star-Rothberger space and

$$\{\mathfrak{U}_n : n \in \omega\}$$

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be a sequence of open covers of $\mathtt{PR}[X]$. Because $\mathtt{PR}[X]$ is a strongly star-Rothberger space, there exists a sequence

$$\{F_n : n \in \omega\}$$

of elements of $\mathtt{PR}[X]$ such that

$$\{\mathtt{St}(F_n, \mathfrak{U}_n) : n \in \omega\}$$

is an open cover of $\mathtt{PR}[X]$. For each $n \in \omega$, we put

$$\mathcal{A}_n = \{F_n\}.$$

Then, $\{\mathcal{A}_n : n \in \omega\}$ is a sequence of finite subsets of $\mathtt{PR}[X]$ and

$$\{\mathtt{St}(\mathcal{A}_n, \mathfrak{U}_n) : n \in \omega\}$$

is an open cover of $\mathtt{PR}[X]$. Therefore, $\mathtt{PR}[X]$ is a strongly star-Menger space.

(d) $\Rightarrow$ (a). Let $\mathtt{PR}[X]$ be a strongly star-Menger space. For each $n \in \omega$, we put

$$\mathfrak{U}_n = \{[\{x\}, X] : x \in X\}.$$

Then, $\{\mathfrak{U}_n : n \in \omega\}$ is a sequence of open covers of $\mathtt{PR}[X]$. Because $\mathtt{PR}[X]$ is a strongly star-Menger space, there exists a sequence

$$\{\mathcal{A}_n : n \in \omega\}$$

of finite subsets of $\mathtt{PR}[X]$ such that

$$\{\mathtt{St}(\mathcal{A}_n, \mathfrak{U}_n) : n \in \omega\}$$

is an open of $\mathtt{PR}[X]$. Now, we will prove that X is countable. In fact, let $x \in X$, then

$$\{x\} \in \mathtt{PR}[X].$$

Thus, there exists $n_x \in \omega$ such that

$$\{x\} \in \mathtt{St}(\mathcal{A}_{n_x}, \mathfrak{U}_{n_x}).$$

This implies that there exists $\mathcal{U}_{n_x} \in \mathfrak{U}_{n_x}$ such that

$$\{x\} \in \mathcal{U}_{n_x} \text{ and } \mathcal{U}_{n_x} \cap \mathcal{A}_{n_x} = \emptyset.$$

Since $\mathcal{U}_{n_x} \in \mathfrak{U}_{n_x}$, we have

$$\mathcal{U}_{n_x} = [\{x\}, X],$$

so that

$$[\{x\}, X] \cap \mathcal{A}_{n_x} = \emptyset.$$

Hence, there exists $F_{n_x} \in \mathcal{A}_{n_x}$ such that

$$F \in [\{x\}, X],$$

it implies that $\{x\} \subset F$. On the other hand, we have

$$\begin{aligned} X = \bigcup_{x \in X} \{x\} &\subset \bigcup_{x \in X} F_{n_x} \\ &\subset \bigcup_{x \in X} \left(\bigcup \mathcal{A}_{n_x}\right) \\ &\subset \bigcup_{n \in \omega} \left(\bigcup \mathcal{A}_n\right). \end{aligned}$$

Because $\bigcup_{n \in \omega} (\bigcup \mathcal{A}_n)$ is countable, X is countable. □

References

1. Trieu HTO, Tuyen LQ, Tuyen OV. Starcompact and related spaces on Pixley-Roy hyperspaces. Novi Sad Journal of Mathematics. 2023;

2. Li Z. The quasi-Rothberger property of Pixley-Roy hyperspaces. Filomat. 2023;37(8):2531-2538.

3. Li Z. The Hurewicz separability of Pixley–Roy hyperspaces. Topology Appl. 2023;324:108-355.

4. Mou L, Li P, Lin S. Regular G_δ-diagonals and hyperspaces. Topology Appl. 2021;301:107-530.

5. Kočinac LDR, Tuyen LQ, Tuyen OV. Some results on Pixley-Roy hyperspaces. J Math. 2022;2022:1-8.

6. Tuyen LQ, Tuyen OV. A remark on Pixley-Roy hyperspaces. Novi Sad J Math. 2024;54(1):183-188.

7. Tuyen LQ, Tuyen OV. The σ-point-finite cn-networks (ck-networks) of Pixley-Roy hyperspaces. Mat. Vesnik. 2023;75(2):134-137.

pISSN 1859-1388
eISSN 2615-9678

8. Xuan WF, Song YK. On cellular-Lindelöf spaces. Iranian Math Soc. 2018;44:1485-1491.
9. Tkachuk VV, Wilson RG. Cellular-compact spaces and their applications. Acta Math Hung. 2019;159:674-688.
10. Kočinac LDR. On Star Selection Principles Theory. Axiom. 2023
11. Pixley C, Roy P. Uncompletable Moore spaces. in: Proc. Auburn Topology Conf. 1969;75-85.
12. van Douwen E. The Pixley-Roy topology on spaces of subsets. Set-theoretic topology, Academic Press. New York. 1977;111-134.
13. Tanaka H. Metrizability of Pixley-Roy hyperspaces. Tsukuba J Math. 1983;7(2):299-315.
14. Engelking R. General Topology. Berlin: Heldermann Verlag; 1989.
15. Xuan WF, Song YK. A study on cellular-Lindelöf spaces. Topology Appl. 2019;251:1-9.