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Second main theorem for Holomorphic curves intersecting a Fermat-type hypersurface in projective spacel

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Abstract. By using a classical result of Cartan, we establish a Second Main Theorem type estimate for an algebraically non-degenerate holomorphic curve into n-dimensional complex projective space intersecting a Fermat-type hypersurface, in which the counting functions are truncated to level n. Consequently, we obtain some degeneracy results and a defect relation for holomorphic curves.

Keywords: Nevanlinna theory, Holomorphic curves, Second Main Theorem, Fermat type hypersurfaces.

1 Introduction

In 1925, Rolf Nevanlinna [1] developed the value distribution theory by comparing the growth of a holomorphic map $f : \mathbb{C} \to \mathbb{CP}^1$ with a number of impacts of f on discs with respect to a collection of $q \geq 3$ distinct points in $\mathbb{CP}^1$. The core of Nevanlinna Theory includes two Main Theorems. The First Main Theorem shows that the value of characteristic function $T_f(r, a)$ does not depend on complex number a, which implies that $T_f(r)$ is an upper bound of the counting function. On the other hand, the Second Main Theorem of Nevanlinna shows that $T_f(r)$ is bounded from above by a sum of at least three distinct counting functions with truncated level 1.

In 1933, Henri Cartan [2] extended Nevanlinna's Second Main Theorem for holomorphic curves intersecting hyperplanes in general position in complex projective spaces. The Cartan-Nevanlina Theorem gives us an estimate about the frequency of intersection of a linearly non-degenerate holomorphic curve $f : \mathbb{C} \to \mathbb{CP}^n$ and a family of $q \geq n + 2$ hyperplanes in general position. It is natural that people want to extend this result for a family of hypersurfaces. This task was first completed by Min Ru [3]. He established a Second Main Theorem for an arbitrary $q \geq n + 2$ hypersurfaces in general position. Then there appeared a demand to obtain an estimate for less than $n + 2$ hypersurfaces. In this direction, Huynh, Vu and Xie [4] proved a Second Main Theorem for only one generic hypersurface of a degree large enough. Later, Yang, Shi and Pang [5] proved a Second Main Theorem for an algebraically non-degenerate holomorphic curve intersecting a Fermat-type hypersurface without the level of truncation. Nguyen [6] improved the result of Yang, Shi and Pang [5] by adding level truncation to the counting function.

Continuing in this direction, in the current paper, we obtain a Second Main Theorem for an algebraically non-degenerate holomorphic curve intersecting a Fermat-type hypersurface with a more simple method. Our estimate is established under these conditions.

A holomorphic curve $f\colon \mathbb{C} \to \mathbb{CP}^n$ is called algebraically non-degenerate if the image of f is not contained in any hypersurface. A family of hypersurfaces $D_1, \ldots, D_q$, $q > n+1$ in $\mathbb{CP}^n$ is said to be in general position if any distinct $n+1$ hypersurfaces among them have empty intersection.

Throughout this paper, for non-negatively valued functions $\varphi(r), \psi(r)$ $(r > 0)$, we write

$$\varphi(r) \leq \psi(r) \ \|$$

if there is a Borel subset $E \subset [0, \infty)$ of finite Lebesgue measure such that the above inequality holds outside E.

Theorem 1.1. *(**Main Theorem**) Let $f : \mathbb{C} \to \mathbb{CP}^n$ be an algebraically non-degenerate holomorphic curve. Let s, b, d be positive integers with $(s+bd) - (n+1)(s+bn) > 0$. Let $D_i = \{z \in \mathbb{CP}^n \mid Q_i(z)R_i^d(z) = 0\}, 0 \leq i \leq n$, be hypersurfaces in general position in $\mathbb{CP}^n$. Let*

$$D = \left\{ z \in \mathbb{CP}^n \mid \sum_{i=0}^{n} Q_i(z)R_i^d(z) = 0 \right\},$$

where Q_i are homogeneous polynomials of degree s and R_i are homogeneous polynomials of degree b. Then

$$[(s+bd) - (n+1)(s+bn)]\, T_f(r) \leq N_f^{[n]}(r, D) + o(T_f(r)) \ \| .$$

In fact, by replacing homogeneous polynomials R_i with z_i, we recover the results of Nguyen [6] and Yang, Shi, Pang [5].

All terminologies will be explained in Section 2. The detailed proof of main results will be presented in Section 3. In the last section, we obtain some corollaries of degeneracy and a truncated defect relation by applying the Main Theorem.

2 Preliminaries

Let $f\colon \mathbb{C} \to \mathbb{CP}^n$ be a holomorphic curve. Let $\mathbf{f} = (f_0 : \ldots : f_n)$ be a reduced representation of f, where f_i are holomorphic functions on $\mathbb{C}$ and having no common zero. Set $\|f(z)\| = \max\{|f_0(z)|, \ldots, |f_n(z)|\}$. The characteristic function of f is defined by

$$T_f(r) := \frac{1}{2\pi} \int_0^{2\pi} \log \left\| \mathbf{f}\left(re^{i\theta}\right) \right\| d\theta,$$

where the above definition is independent, up to an additive constant, of the choice of reduced representation of f.

Let D be a hypersurface in $\mathbb{CP}^n$ of degree d. Let Q be the homogeneous polynomial defining D. Suppose $Q(\mathbf{f}) \not\equiv 0$, the proximity function $m_f(r, D)$ of f with respect to D is defined by

$$m_f(r, D) := \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\left\|\mathbf{f}\left(re^{i\theta}\right)\right\|^d \|Q\|}{|Q(\mathbf{f})\left(re^{i\theta}\right)|} d\theta,$$

where $\|Q\|$ is the maximum modulo of the coefficients of $\mathbb{C}$. The above definition is independent, up to an additive constant, of the choice of reduced representation of f.

To define the counting function, let $n_f(r, D)$ be the number of zeros of $Q(\mathbf{f})$ in the disk $\mathbb{D}(0, r)$, counting with multiplicities. The counting function $N_f(r, D)$ is defined by

$$N_f(r, D) = \int_0^r \left(n_f(t, D) - n_f(0, D)\right) \frac{dt}{t} + n_f(0, D) \log r.$$

For a positive integer M, we denote $n_f^{[M]}(r, D)$ as the number of zeros of $Q(\mathbf{f})$ in $\mathbb{D}(0, r)$, counting at most M times. The truncated counting function to level M of f with respect to D is defined by

$$N_f^{[M]}(r, D) = \int_0^r \left(n_f^{[M]}(t, D) - n_f^{[M]}(0, D)\right) \frac{dt}{t} + n_f^{[M]}(0, D) \log r.$$

By using Jensen's formula, we obtain the First Main Theorem.

Theorem 2.1. *[6] Let $f : \mathbb{C} \to \mathbb{CP}^n$ be a holomorphic curve and let D be a hypersurface in $\mathbb{CP}^n$ of degree d. If $f(\mathbb{C}) \not\subset D$, then for every real number r with $0 < r < +\infty$,*

$$d\, T_f(r) = m_f(r, D) + N_f(r, D) + O(1),$$

where $O(1)$ is a constant independent from r. Consequently,

$$N_f(r, D) \leq d\, T_f(r) + O(1).$$

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We define the defects of f with respect to D by

$$\delta_f(D) = 1 - \lim_{r\to\infty} \sup \frac{N_f(r,D)}{d\,T_f(r)}$$

and

$$\delta_f^{[n]}(D) = 1 - \lim_{r\to\infty} \sup \frac{N_f^{[n]}(r,D)}{d\,T_f(r)}.$$

It follows from the First Main Theorem that $0 \le \delta_f^{[n]}(D) \le \delta_f(D) \le 1$. Hence, the counting function is bounded from above by some multiple of characteristic function. In the reverse direction, people try to give an upper bound to the characteristic function by a certain sum of counting functions. These types of estimates are called Second Main Theorems.

In 1933, H. Cartan [2] proved the Second Main Theorem under these conditions. A holomorphic curve $f\colon \mathbb{C} \to \mathbb{CP}^n$ is called linearly non-degenerate if and only if the image of f is not contained in any hyperplane. A family of hyperplanes $H_1,\dots,H_q$, $q > n+1$ in $\mathbb{CP}^n$ is said to be in general position if any distinct $n+1$ hyperplanes among them have empty intersection.

Theorem 2.2. *[2] Let $f\colon \mathbb{C} \to \mathbb{CP}^n$ be a linearly non-degenerate holomorphic curve. Let $\{H_\lambda\}_{\lambda=1,\dots,q}$ be a family of $q \ge n+2$ hyperplanes in $\mathbb{CP}^n$ in general position. Then*

$$(q-n-1)\,T_f(r) \le \sum_{\lambda=1}^{q} N_f^{[n]}(r,H_\lambda) + o(T_f(r)) \quad \|.$$

Example 2.3. Consider the following holomorphic map $f\colon z \in \mathbb{C} \to (1 : e^z : e^{2z}) = (w_0 : w_1 : w_2) \in \mathbb{CP}^2$ and the following four lines on $\mathbb{CP}^2$ given by
$L_i = \{w_i = 0\}\,(0 \le i \le 2)$, $\quad L_3 = \{w_0 + w_1 + w_2 = 0\}$.

The characteristic function of f is calculated as follows:

$$\begin{aligned}
T_f(r) &= \frac{1}{4\pi}\int_{|z|=r} \log(1+|e^z|^2+|e^{2z}|^2)d\theta + O(1)\\
&= \frac{1}{4\pi}\int_{|z|=r} \log(1+e^{2r\cos\theta}+e^{4r\cos\theta})d\theta + O(1)\\
&= \frac{1}{4\pi}\int_{\cos\theta>0} \log(1+e^{2r\cos\theta}+e^{4r\cos\theta})d\theta + O(1)\\
&= \frac{1}{4\pi}\int_{-\pi/2}^{\pi/2} 4r\cos\theta d\theta + O(1)\\
&= \frac{2r}{\pi} + O(1).
\end{aligned}$$

Besides, $f(\mathbb{C}) \cap L_i = \emptyset$ for $0 \le i \le 2$, thus $N_f(r,L_i) = 0\,(0 \le i \le 2)$. By the First Main Theorem, one has $m_f(r,L_i) + N_f(r,L_i) = T_f(r)$, so $m_f(r,L_i) = T_f(r) = \frac{2r}{\pi} + O(1)$ $(0 \le i \le 2)$. Now, let us compute the counting function and the proximity function of f with respect to L_3. Using Cartan's Second Main Theorem for f and the family $\{L_i\}_{0\le i\le 3}$, one has

$$\begin{aligned}
T_f(r) &\le \sum_{i=0}^{3} N_f^{[2]}(r,L_i) + o(T_f(r)) \quad \|\\
&\le N_f(r,L_3) + o(T_f(r)) \quad \|\,.
\end{aligned}$$

Combining this with the First Main Theorem, we have

$$N_f(r,L_3) = T_f(r) + o(T_f(r)) = \frac{2r}{\pi} + o(r)\ \|,$$

and hence $m_f(r,L_3) = o(r)\ \|$.

3 Proof of the Main Theorem

Before giving proof for the Main Theorem, we claim the following lemma.

Lemma 3.1. *Let $f : \mathbb{C} \to \mathbb{CP}^n$ be a holomorphic curve. Let*

$$D_i = \{z \in \mathbb{CP}^n \mid Q_i(z) = 0\}\,, \quad 0 \le i \le n,$$

be hypersurfaces of degree d in general position. Suppose that the image of f is not contained in any D_i. Let π be the map given as

$$\begin{aligned}
\pi :\quad &\mathbb{CP}^n \longrightarrow \mathbb{CP}^n\\
Z := [z_0 : z_1 : \ldots : z_n] &\longmapsto [Q_0(Z) : Q_1(Z) : \ldots : Q_n(Z)]\,.
\end{aligned}$$

Then, for $g = \pi \circ f\colon \mathbb{C} \to \mathbb{CP}^n$, one has

$$T_g(r) = d\,T_f(r) + O(1).$$

Proof. Consider the set $E = \{z \in \mathbb{C}^{n+1} \setminus \{0\} \mid \|Z\| = 1\}$. Since E is compact, for every Z in E, there exist two positive numbers C_1, C_2 such that

$$C_1 \le \max_{0\le i\le n} |Q_i(Z)| \le C_2, \forall z \in E.$$

For every $Z \in \mathbb{C}^{n+1}\setminus\{0\}$, one has $\left\| \frac{Z}{\|Z\|} \right\| = 1$. Thus, $\frac{Z}{\|Z\|} \in E$. Hence, $C_1 \leq \max_{0\leq i\leq n} \left| Q_i\left(\frac{Z}{\|Z\|}\right)\right| \leq C_2$.

Since Q_i are homogeneous polynomials of degree d, the above inequality implies

$$C_1 \|Z\|^d \leq \|[Q_0(Z) : Q_1(Z) : \ldots : Q_n(Z)]\| \leq C_2 \|Z\|^d .$$

From the above inequalities, one has

$$C_1 \|f\|^d \leq \|g\| \leq C_2 \|f\|^d .$$

Then, by the definition of Characteristic function, we obtain

$$T_g(r) = d\, T_f(r) + O(1).$$

Now, with the use of Lemma 3.1, we can establish a proof for the Main Theorem. □

Proof of Theorem 1.1

Let π be a map as follows:

$$\pi : \quad \mathbb{CP}^n \longrightarrow \mathbb{CP}^n$$
$$[z_0 : z_1 : \ldots : z_n] \longmapsto \left[Q_0R_0^d : Q_1R_1^d : \ldots : Q_nR_n^d\right].$$

Consider the curve $g = \pi \circ f$. Since f is algebraically non-degenerate, g is linearly non-degenerate. Let

$$H_i = \{z_i = 0\}, 0 \leq i \leq n,$$
$$H_{n+1} = \left\{\sum_{i=0}^{n} z_i = 0\right\}.$$

These $n+2$ hyperplanes are in general position. Hence, by applying Cartan's Second Main Theorem, we obtain

$$T_g(r) \leq \sum_{i=0}^{n} N_g^{[n]}(r, H_i) + N_g^{[n]}(r, H_{n+1}) + o(T_g(r)) \parallel . \tag{3.1}$$

By applying the First Main Theorem and Lemma 3.1, we get some estimates

$$T_g(r) = (s+bd)\, T_f(r) + O(1), \tag{3.2}$$
$$N_g^{[n]}(r, H_i) \leq N_f^{[n]}(r, \{Q_i = 0\}) + N_f^{[n]}(r, \{R_i^d = 0\})$$
$$\leq N_f(r, \{Q_i = 0\}) + \frac{n}{d} N_f(r, \{R_i^d = 0\})$$
$$\leq s\, T_f(r) + bn\, T_f(r, \{R_i^d = 0\}), \tag{3.3}$$
$$N_g^{[n]}(r, H_{n+1}) = N_f^{[n]}(r, D). \tag{3.4}$$

Then, by combining (3.1), (3.2), (3.3), (3.4), we obtain

$$(s+bd)\, T_f(r) \leq (n+1)(s+bn)\, T_f(r) + N_f^{[n]}(r, D) \tag{3.5}$$
$$+ o(T_f(r)) \parallel, \tag{3.6}$$

which implies

$$[(s+bd) - (n+1)(s+bn)]\, T_f(r) \leq N_f^{[n]}(r, D) + o(T_f(r)) \parallel .$$

Remark 3.2. The inequality is meaningful if and only if the left-hand side is greater than zero, which means $(s+bd) - (n+1)(s+bn) > 0$.

4 Some corollaries

From the Main Theorem, we establish some corollaries about the algebraic degeneracy of entire curves.

Corollary 4.1. *Let s, b, d be positive integers such that $(s+bd) - (n+1)(s+bn) > 0$. Let D be the same hypersurface in Theorem 1.1. Then, every holomorphic curve $f : \mathbb{C} \to \mathbb{CP}^n\setminus D$ must be algebraically degenerate.*

Proof. Suppose f is algebraically non-degenerate. By applying Theorem 1.1 for f and D, we have

$$[(s+bd) - (n+1)(s+bn)]\, T_f(r) \leq N_f^{[n]}(r, D) + o(T_f(r)) \parallel .$$

Since f misses D, one has $N_f^{[n]}(r, D) = 0$. Hence,

$$[(s+bd) - (n+1)(s+bn)]\, T_f(r) \leq o(T_f(r)) \parallel .$$

This is a contradiction since

$$(s+bd) - (n+1)(s+bn) > 0.$$

So, f is algebraically degenerate. □

Corollary 4.2. *Let s, b, d be positive integers such that $(s+bd) - (n+1)(s+bn) > 0$. Let D be the same hypersurface in Theorem 1.1. Then, every holomorphic curve $f : \mathbb{C} \to \mathbb{CP}^n$ whose image intersects D with multiplicity at least $l > \frac{n(s+bd)}{(s+bd) - (n+1)(s+bn)}$ must be algebraically degenerate.*

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Proof. Suppose f is algebraically non-degenerate. By applying Theorem 1.1 and the First Main Theorem for f and D, we have

$$\begin{aligned}&[(s+bd)-(n+1)(s+bn)]\,T_f(r)\\&\le N_f^{[n]}(r,D)+o(T_f(r))\ \|\\&\le n\,N_f^{[1]}(r,D)+o(T_f(r))\ \|\\&\le \frac{n}{l}\,N_f(r,D)+o(T_f(r))\ \|\\&\le \frac{n(s+bd)}{l}\,T_f(r)+o(T_f(r))\ \|\ .\end{aligned}$$

This implies

$$\left(\frac{(s+bd)-(n+1)(s+bn)}{n(s+bd)}-\frac{1}{l}\right)T_f(r)\le o(T_f(r))\ \|,$$

which gives a contradiction with

$$l>\frac{n(s+bd)}{(s+bd)-(n+1)(s+bn)}.$$

Therefore, f must be algebraically degenerate. □

Corollary 4.3. *Let s,b,d be positive integers such that $(s+bd)-(n+1)(s+bn)>0$. Let D be the same hypersurface in Theorem 1.1. Then, we have the following **truncated defect relation***

$$\delta_f^{[n]}(D)\le\frac{(n+1)(s+bn)}{s+bd}.$$

Proof. First, we rewrite Theorem 1.1 as follows:

$$\begin{aligned}(s+bd)\,T_f(r)-N_f^{[n]}(r,D)\le{}&(n+1)(s+bn)\,T_f(r)\\&+o(T_f(r))\ \|\ .\end{aligned}$$

Then, by dividing both sides of the above inequality by $(s+bd)\,T_f(r)$, one has

$$1-\frac{N_f^{[n]}(r,D)}{(s+bd)\,T_f(r)}\le\frac{(n+1)(s+bn)}{s+bd}+\frac{o(T_f(r))}{(s+bd)\,T_f(r)}\ \|\ .$$

Now, let $r\to\infty$, one has

$$\begin{aligned}1-\limsup_{r\to\infty}\frac{N_f^{[n]}(r,D)}{d\,T_f(r)}\le{}&\liminf_{r\to\infty}\frac{(n+1)(s+bn)}{s+bd}\\&+\liminf_{r\to\infty}\frac{o(T_f(r))}{(s+bd)\,T_f(r)}.\end{aligned}$$

Since $\liminf\limits_{r\to\infty}\dfrac{o(T_f(r))}{(s+bd)\,T_f(r)}=0$, we obtain

$$\delta_f^{[n]}(D)\le\frac{(n+1)(s+bn)}{s+bd}.$$

□

Example 4.1. Consider the following holomorphic map $f: z\in\mathbb{C}\to(1:e^z:e^{cz})=(w_0:w_1:w_2)\in\mathbb{CP}^2$, where $c>1$ is an irrational number and D is the algebraic curve defined as

$$Q_0w_0^d+Q_1w_1^d+Q_2w_2^d=0,$$

where $d>10$ and $Q_i\,(0\le i\le 2)$ are three conics. Suppose that the family $\{Q_0,Q_1,Q_2,L_0,L_1,L_2\}$ is in general position, where $L_i=\{w_i=0\}\,(0\le i\le 2)$. By similar computations as in Example 2.3, we have $T_f(r)=\frac{cr}{\pi}$. Since c is irrational, f is algebraically non-degenerate. Hence, the Main Theorem yields: $(d-10)\,T_f(r)\le N_f^{[2]}(r,D)+o(T_f(r))\ \|$, which implies that $\frac{(d-10)cr}{\pi}\le N_f^{[2]}(r,D)+o(r)\ \|$. By Corollary 4.3, we have the defect relation $\delta_f^{[2]}(D)\le\frac{12}{d+2}$. Furthermore, applying Corollaries 4.1, 4.2, we see that

(a) For any integer number $d>10$, all holomorphic curve $g\colon\mathbb{C}\to\mathbb{CP}^2\setminus D$ must be algebraically degenerate;

(b) For arbitrary two integer numbers d,ℓ with $d>10$ and $\ell>\frac{2(d+2)}{d-10}$, all holomorphic curve $g\colon\mathbb{C}\to\mathbb{CP}^2$ whose image intersects D with multiplicity at least ℓ must be algebraically degenerate.

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