

Using complex scaling method based on Jost function formalism to investigate single-particle resonances in ^{24}O

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Abstract. We apply the complex scaling method (CSM) to study single-particle resonance states in ^{24}O by using the Jost function formalism with a Woods-Saxon plus spin-orbit potential. The CSM transformation $r \rightarrow re^{i\theta}$ rotates the branch cut of the complex momentum plane, enabling direct location of resonance poles on the physical Riemann sheet without analytic continuation. The complex-scaled Jost function $J_{lj}^{(+)\theta}(k)$ is evaluated numerically, and its zeros are found with the Newton-Raphson method. A scan over five rotation angles $\theta = n\theta_M$ ($n = 1, \dots, 5$; $\theta_M = 0.07$ rad is the scan step) confirms the θ -independence of resonance pole positions, as required by the Aguilar-Balslev-Combes theorem. The physical resonance energy $E_{\text{res}} = \frac{\hbar^2}{2m}(k^\theta e^{-i\theta})^2$ is θ -independent for $\theta > \theta_c$. Four single-particle resonances above the neutron threshold are identified for the $f_{7/2}, f_{5/2}, g_{9/2}$ and $g_{7/2}$ partial waves, with resonance energies of $E_r \approx 1.85, 7.0, 11.2$ and 17.5 MeV, respectively.

Keywords: complex scaling method, Jost function, resonance, ^{24}O , Woods-Saxon potential

1 Introduction

Neutron-rich oxygen isotopes near the neutron drip line occupy a central position in modern nuclear structure research. ^{24}O is one of the heaviest bound oxygen isotopes, and its single-particle spectrum—including both bound and unbound (resonance) states—is essential for interpreting experimental data and constraining theoretical models [1–3]. Beyond ^{24}O , the next isotope ^{25}O is unbound, making the resonances of the neutron ^{24}O system directly relevant to understanding the oxygen drip line.

In the standard scattering approach, resonance states appear as zeros of the outgoing Jost function $J_{lj}^{(+)\theta}(k) = 0$ at $\text{Im}(k) < 0$, on the unphysical (second) Riemann sheet [4, 5]. Here the two sheets are defined by the sign of $\text{Im}(k)$: the physical (first) sheet has $\text{Im}(k) \geq 0$, and the unphysical (second) sheet has $\text{Im}(k) < 0$.

Locating these poles requires analytic continuation to the unphysical Riemann sheet, which is challenging, particularly for narrow resonances that produce very sharp structures in the complex plane.

The Complex Scaling Method (CSM) [6–8] provides an elegant alternative. Under $r \rightarrow re^{i\theta}$, the branch cut rotates by $-\theta$, and resonance poles that were hidden in the lower half k -plane at $\theta = 0$ move into the upper half-plane when θ exceeds a critical angle θ_c . The Aguilar-Balslev-Combes (ABC) theorem [9, 10] guarantees the θ -independence of resonance energies, providing a rigorous criterion for distinguishing genuine resonances from non-resonant continuum states. The CSM further makes the resonance wave function square-integrable at $\theta > \theta_c$, a property that is impossible at $\theta = 0$, where the wave function grows exponentially at large r [11]. Such resonant states, together with bound states and the non-

resonant continuum, form the Berggren basis in the complex energy plane, which has been used to describe two-particle resonant states [12] and weakly bound neutron-rich nuclei within the Gamow shell model [13].

In this work we implement the CSM within the Jost-function framework [14, 15] and apply it to ^{24}O , identifying four single-particle resonances ($f_{7/2}, f_{5/2}, g_{9/2}, g_{7/2}$) using three mutually consistent observables: the T-matrix peak, the level density maximum, and the Newton-Raphson pole position. We also compute the regular wave functions at each resonance energy, demonstrating directly the transition from a diverging wave function at $\theta = 0$ to a spatially compact, square-integrable one at $\theta = 0.35$ rad. Section 2 develops the theoretical framework. Section 3 presents the numerical results. Conclusions are in Section 4.

2 Theoretical framework

2.1 Complex scaling method

The complex scaling method rotates the complex energy/momentum plane and changes the branch cut while keeping kr invariant. The transformation of the radial coordinate and complex momentum is

$$r' = e^{i\theta}r, \quad k' = e^{-i\theta}k \quad (1)$$

where θ is the rotation angle. The complex scaling operator $\hat{U}(\theta)$ is defined by $\hat{U}(\theta)f(r) = f(re^{i\theta})$ and satisfies $\hat{U}(\theta_1)\hat{U}(\theta_2) = \hat{U}(\theta_1 + \theta_2)$, $\hat{U}^{-1}(\theta) = \hat{U}(-\theta)$. Solving $d\hat{U}/d\theta = (ir\partial/\partial r - ik\partial/\partial k)\hat{U}$ gives

$$\hat{U}(\theta) = \exp\left(ir\theta \frac{\partial}{\partial r}\right) \cdot \exp\left(-ik\theta \frac{\partial}{\partial k}\right) \quad (2)$$

2.2 Complex scaled Schrödinger equation

The single-particle Schrödinger equation with potential $V_{lj}(r)$ is

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} - \frac{l(l+1)}{r^2}\right) + V_{lj}(r)\right]\varphi_{lj} = \frac{\hbar^2 k^2}{2m}\varphi_{lj} \quad (3)$$

Applying $\hat{U}(\theta)$ and using the Baker-Campbell-Hausdorff expansion, we transform the kinetic term as $\hat{U}\partial^2/\partial r^2\hat{U}^{-1} = e^{-2i\theta}\partial^2/\partial r^2$, and the potential as $\hat{U}(\theta)V_{lj}(r)\hat{U}^{-1}(\theta) = V_{lj}(re^{i\theta})$, yielding

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} - \frac{l(l+1)}{r^2}\right) + W_{lj}^\theta(r)\right]\varphi_{lj}^\theta = \frac{\hbar^2 k^2}{2m}\varphi_{lj}^\theta \quad (4)$$

where $\varphi_{lj}^\theta(r; k) \equiv \hat{U}(\theta)\varphi_{lj}(r; k)$, and the complex-scaled potential is

$$W_{lj}^\theta(r) \equiv e^{2i\theta}V_{lj}(re^{i\theta}) \quad (5)$$

Even for a real potential V_{lj} , the scaled potential W_{lj}^θ is complex for $\theta \neq 0$, since the argument $re^{i\theta}$ is complex. This non-Hermitian character allows resonance poles to appear on the physical Riemann sheet.

2.3 Complex-scaled Jost function

Let $u_{lj}^\theta(r; k)$ and $v_{lj}^{(\pm)\theta}(r; k)$ be the regular and irregular solutions of Eq. (4). The complex-scaled Jost function $J_{lj}^{(\pm)\theta}(k)$ is defined as the coefficient in the expansion [5, 14]

$$u_{lj}^\theta = \frac{1}{2}\left[J_{lj}^{(+)\theta}(k)v_{lj}^{(-)\theta} + J_{lj}^{(-)\theta}(k)v_{lj}^{(+)\theta}\right] \quad (6)$$

and is evaluated numerically as the Wronskian

$$J_{lj}^{(\pm)\theta}(k) = \pm \frac{k}{i}\left[u_{lj}^\theta \frac{\partial v_{lj}^{(\pm)\theta}}{\partial r} - v_{lj}^{(\pm)\theta} \frac{\partial u_{lj}^\theta}{\partial r}\right] \quad (7)$$

Note that $J_{lj}^{(\pm)\theta}(k)$ is the complex-scaled Jost function, defined with the scaled potential W_{lj}^θ and complex-scaled irregular solutions $v_{lj}^{(\pm)\theta}$. When $\theta = 0$, it reduces to the standard Jost function $J_{lj}^{(\pm)\theta}(k)$, whose zeros at $\text{Im}(k) < 0$ are the resonance poles on the unphysical sheet. When $\theta > 0$, $J_{lj}^{(\pm)\theta}(k)$ is a distinct analytic function whose zeros at $\text{Im}(k) > 0$ encode the same resonances—now on the physical sheet, without analytic continuation.

A resonance pole at $k_{res}(\theta = 0) = k_r - ik_i$, ($k_r, k_i > 0$) is rotated to the upper half k -plane once

$$\theta > \theta_c = \arctan\left(\frac{k_l}{k_r}\right) \quad (8)$$

Regarding the ABC theorem [9, 6], the resonance energy $E_{res} = E_r - i\Gamma/2$ is independent of θ for $\theta > \theta_c$, while non-resonant continuum poles rotate along the -2θ ray in the complex E-plane. The physical resonance energy is recovered from the CSM wavenumber k^θ via

$$E_{res} = \frac{\hbar^2}{2m} (k^\theta e^{-i\theta})^2 \quad (9)$$

which maps k^θ back to the physical sheet.

2.4 S-Matrix, T-Matrix, level density, and bound states

After applying the CSM, we expressed the S-matrix and T-matrix through the complex-scaled Jost functions [16] as

$$S_{lj}^\theta(k) = \frac{J_{lj}^{(-)\theta}(k)}{J_{lj}^{(+)\theta}(k)} \quad (10)$$

$$T_{lj}^\theta(k) = \frac{i}{2} \left[\frac{J_{lj}^{(-)\theta}(k)}{J_{lj}^{(+)\theta}(k)} - 1 \right] \quad (11)$$

The single-particle level density is

$$g(E) = -\frac{1}{\pi} \Sigma_{lj} (2j+1) \text{Im} \left[\frac{d}{dE} \ln J_{lj}^{(+)\theta}(k(E)) \right] \quad (12)$$

where the derivative is evaluated by the chain rule $d/dE = (dk/dE) d/dk$, with $k(E) = \sqrt{2mE}/\hbar$ on the physical sheet ($\text{Im}(k) \geq 0$). For bound states, $J_{lj}^{(+)\theta}(k_n) = 0$ when $k_n = i\kappa_n$ ($\kappa_n > 0$) and the radial wave function is obtained from Eq. (6):

$$u_{lj}^\theta(r; k_n) \propto J_{lj}^{(-)\theta}(k_n) v_{lj}^{(+)\theta}(r; k_n) \quad (13)$$

Bound-state energies are real and θ -independent, which is consistent with the ABC theorem.

3 Numerical calculation and discussion

3.1 Potential parameters

The nucleon-nucleus interaction is described as a Woods-Saxon (WS) plus spin-orbit potential

$$V_{lj}(r) = \left(1 - 0.67 \tau_z \frac{N - Z}{A} \right) \times \left[V_1 f_{WS}(x_1) + V_4 (\ell \cdot \sigma) \frac{g_{WS}(x_1)}{r a_1} \right] \quad (14)$$

with

$$f_{WS}(x) = \frac{1}{1 + e^x}, \quad g_{WS}(x) = \frac{e^x}{(1 + e^x)^2} \quad (15)$$

$x_i = (r - R_i)/a_i$, $R_i = \alpha_i A^{r_i} + \beta_i$, and $\ell \cdot \sigma = j(j+1) - l(l+1) - 3/4$. For neutrons, $\tau_z = +1$. Table 1 gives the parameters for neutron+ ^{24}O ($A = 24, N = 16, Z = 8$).

Table 1. WS parameters for neutron+ ^{24}O ($R_i = \alpha_i A^{1/3} \text{ fm}$)

Term	V (MeV)	a (fm)	α (fm)	β (fm)	γ	R (fm)
Central	-60.0	0.67	1.27	0.0	1/3	3.66
Spin-orbit	-17.0	0.67	1.27	0.0	1/3	3.66

Eq. (4) is solved with the Numerov method on a radial mesh $r \in [0, 20]$ fm with step $\Delta r = 0.1$ fm ($N = 200$ points). The Jost function is evaluated as the Wronskian at matching radius $r_0 = 10$ fm. Nine partial waves from $s_{1/2}$ to $g_{9/2}$ are included. Resonance poles are located via Newton-Raphson minimization of $|J_{lj}^{(\pm)\theta}(k)|$.

3.2 Bound states

Table 2 lists the single-particle bound states of neutron+ ^{24}O found by bisection at $\theta = 0$. Six bound states are obtained: $1s_{1/2}$ through $1d_{3/2}$, with the shallowest being $1d_{3/2}$ when $E = -3.64$ MeV. All $l \geq 3$ (f and g) orbits lie above the neutron threshold and appear as resonances. These energies are θ -independent, which is in agreement with the ABC theorem. It should be noted that the global Woods-Saxon parameters of Table 1 place $1d_{3/2}$ about 4.4 MeV below its measured position: in ^{25}O this orbital is unbound and appears as a resonance at 0.75 MeV [17]. Such a shift is a known limitation of global Woods-

Saxon parametrizations near the neutron drip line, and the single-particle energies obtained here accordingly characterize the potential of Table 1 rather than constituting predictions for ^{25}O .

Table 2. Bound states of neutron+ ^{24}O ($\theta = 0$)

State	E _n (MeV)
$1s_{1/2}$	-31.07
$1p_{3/2}$	-20.06
$1p_{1/2}$	-17.06
$1d_{5/2}$	-8.75
$2s_{1/2}$	-6.28
$1d_{3/2}$	-3.64

3.3 Complex scaled potential

Fig. 1 shows the effective potential $W_{lj}^\theta(r)$ for $g_{9/2}$ when $\theta = 0$ and $\theta = 0.35$ rad.

When $\theta = 0$, the potential is purely real: a deep attractive WS well ($V_1 = -60$ MeV) combined with a repulsive centrifugal barrier, creating a potential pocket that supports quasi-bound states.

When $\theta = 0.35$ rad, a significant imaginary component develops near the nuclear surface ($r \approx 4$ fm). This arises from the analytic continuation $V_{lj}(re^{i\theta})$ of the real WS potential. The real part of W^θ remains comparable in magnitude with the $\theta = 0$ case, while the surface imaginary part provides the non-Hermitian coupling that rotates resonance poles into the upper half k -plane. The imaginary potential is steepest near $r = R$, closely related to the slope of the WS form factor at the nuclear surface.

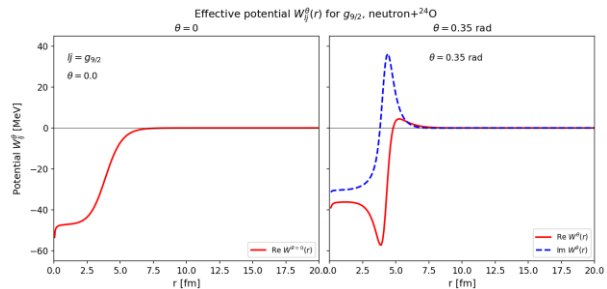


Fig. 1. Effective radial potential $W_{lj}^\theta(r)$ for $g_{9/2}$ when $\theta = 0$ (left, purely real) and $\theta = 0.35$ rad (right). Red solid: $\text{Re}[W^\theta]$; blue dashed: $\text{Im}[W^\theta]$

3.4 Resonance pole trajectory

Fig. 2 shows the resonance pole positions on the complex E -plane when $\theta = n\theta_M$ ($n = 1, \dots, 5$; $\theta_M = 0.07$ rad). The four physical resonances are listed in Table 3.

(i) *θ -independence of $\text{Re}(E_{\text{pole}})$.* For each partial wave, $\text{Re}(E_{\text{pole}})$ is approximately constant across all five θ steps—the hallmark of a genuine physical resonance (ABC theorem).

(ii) *Continuum states sweep.* Non-resonant continuum poles follow the -2θ rays in Fig. 2, moving rapidly when θ increases and easily distinguished from the stationary resonance poles.

(iii) *Crossing the real axis.* For $f_{5/2}$ and $g_{9/2}$, the pole crosses from $\text{Im}(E) < 0$ when $\theta = 0.07$ rad to $\text{Im}(E) > 0$ when $\theta = 0.14$ rad, giving $\theta_c \approx 0.10 - 0.11$ rad. For $g_{7/2}$, crossing occurs around $\theta \approx 0.22$ rad.

(iv) *Very narrow $f_{7/2}$ resonance.* The $f_{7/2}$ pole is already at $\text{Im}(E) > 0$ when $\theta = 0.07$ rad, implying $\theta_c < 0.07$ rad and a very small width $\Gamma \ll 1$ MeV, reflecting the strong $l = 3$ centrifugal barrier.

Table 3. Physical resonance energies E_r and widths Γ (MeV) vs. θ , obtained from Eq. (9). Entries show (E_r, Γ) . $\checkmark$ denotes $\theta > \theta_c$ (ABC-stable)

θ (rad)	$f_{5/2}$	$f_{7/2}$	$g_{9/2}$	$g_{7/2}$
	$(E_r \text{ [MeV]}, \Gamma \text{ [MeV]})$	$(E_r \text{ [MeV]}, \Gamma \text{ [MeV]})$	$(E_r \text{ [MeV]}, \Gamma \text{ [MeV]})$	$(E_r \text{ [MeV]}, \Gamma \text{ [MeV]})$

0.07	(7.06, 3.22)	(1.85, 0.09)✓	(11.72, 5.56)	(19.88, 10.75)
0.14	(7.04, 3.50)✓	(1.85, 0.09)✓	(11.25, 4.96)✓	(19.27, 13.90)
0.21	(7.03, 3.70)✓	(1.85, 0.09)✓	(11.23, 4.75)✓	(18.49, 16.89)
0.28	(7.02, 3.78)✓	(1.85, 0.09)✓	(11.23, 4.71)✓	(17.72, 18.75)✓
0.35	(7.02, 3.81)✓	(1.85, 0.09)✓	(11.23, 4.71)✓	(17.36, 19.12)✓
θ_c	≈ 0.11	< 0.07	≈ 0.11	≈ 0.23

The resonance energies and widths obtained from Eq. (9), averaged over the ABC-stable θ steps, are as follows: $f_{7/2}$: $E_r \approx 1.85$ MeV, $\Gamma \approx 0.09$ MeV (all five steps); $f_{5/2}$: $E_r \approx 7.02$ MeV, $\Gamma \approx 3.7$ MeV ($\theta \geq 0.14$ rad); $g_{9/2}$: $E_r \approx 11.2$ MeV, $\Gamma \approx 4.8$ MeV ($\theta \geq 0.14$ rad); $g_{7/2}$: $E_r \approx 17.5$ MeV, $\Gamma \approx 19$ MeV ($\theta = 0.28$ and 0.35 rad). The widths increase with energy and angular momentum l , which is consistent with decreasing centrifugal barrier height.

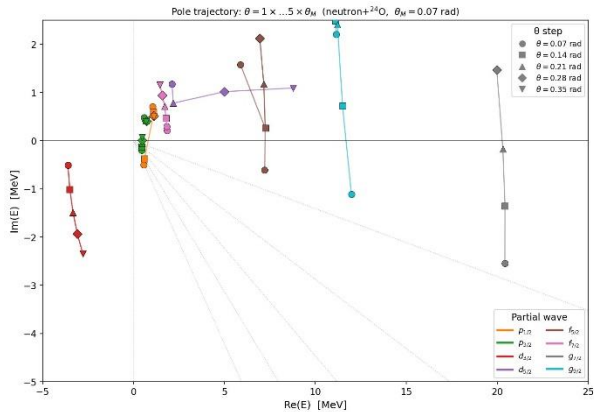


Fig. 2. Resonance pole positions on the complex E-plane for $\theta = n\theta_M$ ($n = 1, \dots, 5$; $\theta_M = 0.07$ rad). Markers identify $n = 1, \dots, 5$. Colors identify partial waves.

Dotted lines: continuum cut rays $Im(E) = -Re(E) \tan(2n\theta_M)$

3.5 T-Matrix and level density for all resonance channels

Figs. 3–6 present the two-panel analysis for all four resonance partial waves when $\theta = 0.35$ rad: (a) T-matrix $|T_{lj}^\theta(E)|^2$ and (b) level density $g(E)$. Both panels share the horizontal axis $Re(E) \in [0, 30]$ MeV. The vertical red dashed line marks E_r from the Newton-Raphson pole search via Eq. (9). In each case, the T-matrix peak, level density maximum, and pole position are mutually

consistent, providing three independent confirmations of the resonance identification.

$f_{5/2}$ channel (Fig. 3). A broad resonance, when $E_r \approx 7.02$ MeV with $\Gamma \approx 3.81$ MeV, appears as a wide maximum in $|T|^2$ and a broad bump in $g(E)$. The width reflects the intermediate strength of the $l = 3$ centrifugal barrier combined with the spin-orbit splitting: the $f_{5/2}$ orbit lies at $\Delta E \approx 5$ MeV above the $f_{7/2}$ state, where the barrier is less effective at confining the resonance. The Newton-Raphson pole, when $\theta = 0.35$ rad, gives $E_r = 7.02$ MeV, consistent with the T-matrix peak.

$f_{7/2}$ channel (Fig. 4). A sharp peak in $|T|^2$ at $E_r \approx 1.85$ MeV is accompanied by a pronounced spike in $g(E)$ at the same energy. The coincidence of the T-matrix peak, level density spike, and Newton-Raphson pole ($E_r = 1.85$ MeV, $\Gamma = 0.09$ MeV, all five θ steps) confirms the consistent identification of the same physical resonance through three independent observables. The very narrow width reflects the strong $l = 3$ centrifugal barrier. This state has $\theta_c < 0.07$ rad—the smallest critical angle of all four resonances.

$g_{9/2}$ channel (Fig. 5). A broad resonance at $E_r \approx 11.23$ MeV ($\Gamma \approx 4.71$ MeV when $\theta = 0.35$ rad) appears as a wide maximum in $|T|^2$ and a broad bump in $g(E)$. The Newton-Raphson pole at $\theta = 0.35$ rad gives $E_r = 11.23$ MeV, in excellent agreement with the T-matrix and $g(E)$ peak positions, validating the CSM implementation.

$g_{7/2}$ channel (Fig. 6). A very broad resonance is centered near $E_r \approx 18$ MeV. Due to its large width ($\Gamma \approx 18$ MeV when $\theta = 0.35$ rad), the T-matrix peak and $g(E)$ bump are broad, and the resonance structure merges with the background. The critical angle $\theta_c \approx 0.23$ rad is

the largest of all four resonances; two steps ($\theta = 0.28$ and 0.35 rad) are in the threshold. A wider angular scan would be needed to obtain a converged Γ for this state.

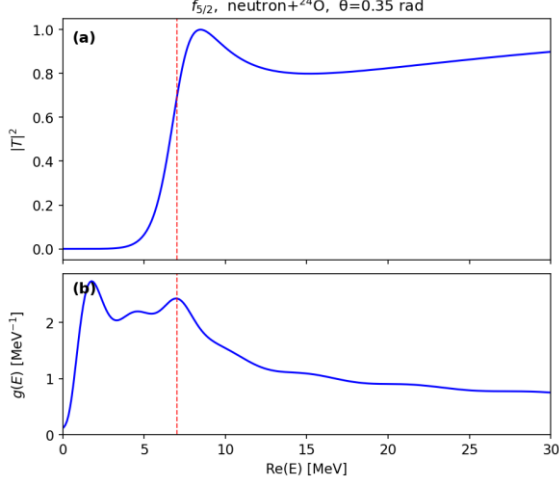


Fig. 3. Two-panel analysis for $f_{5/2}$, $\theta = 0.35$ rad: (a) $|T_{ij}^\theta(E)|^2$; (b) level density $g(E)$ [MeV $^{-1}$]. Red dashed line: $E_r = 7.02$ MeV from Eq. (9)

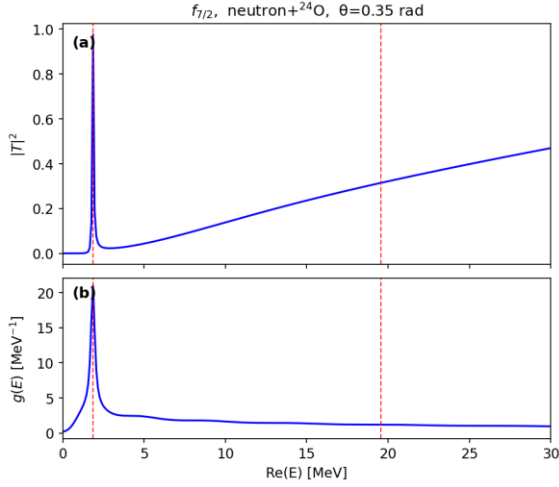


Fig. 4. Same as Fig. 3 but for $f_{7/2}$. Red dashed line: $E_r = 1.85$ MeV. Very narrow width $\Gamma \approx 0.09$ MeV reflects the strong centrifugal barrier

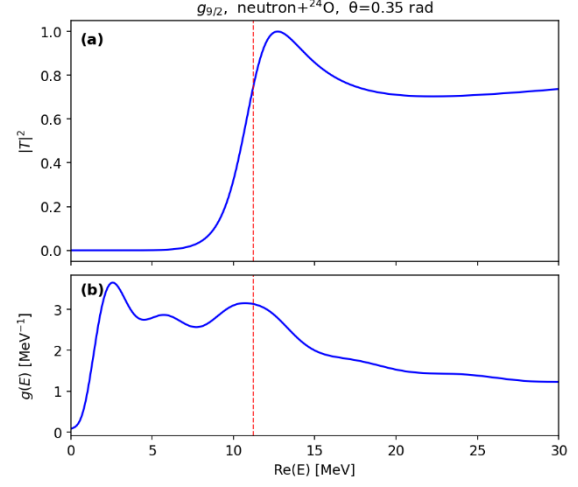


Fig. 5. Same as Fig. 3 but for $g_{9/2}$. Red dashed line: $E_r = 11.23$ MeV. Broad width $\Gamma \approx 4.71$ MeV reflects weaker centrifugal barrier for $l = 4$

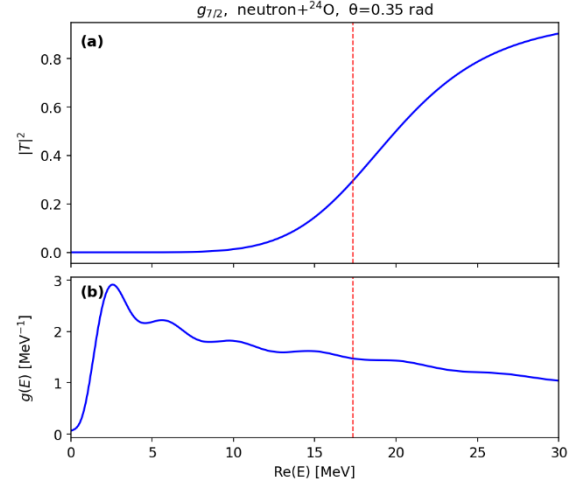


Fig. 6. Same as Fig. 3 but for $g_{7/2}$. Red dashed line: $E_r \approx 17.36$ MeV. Very large width $\Gamma \approx 18$ MeV makes the resonance structure merge with the background

3.6 Regular wave functions at resonance

Fig. 7 shows the regular wave functions $u_{ij}^\theta(r; k_{res})$ at the resonance momentum for all four partial waves, comparing $\theta = 0$ (unscaled, red) and $\theta = 0.35$ rad (CSM-scaled, blue). Real parts are solid lines; imaginary parts are dashed. All wave functions are normalised via the peak amplitude of the $\theta = 0.35$ rad solution.

Regular solution $u_{ij}^\theta(r)$ at the resonance pole (neutron+ ^{24}O , $\theta_M = 0.07$ rad)

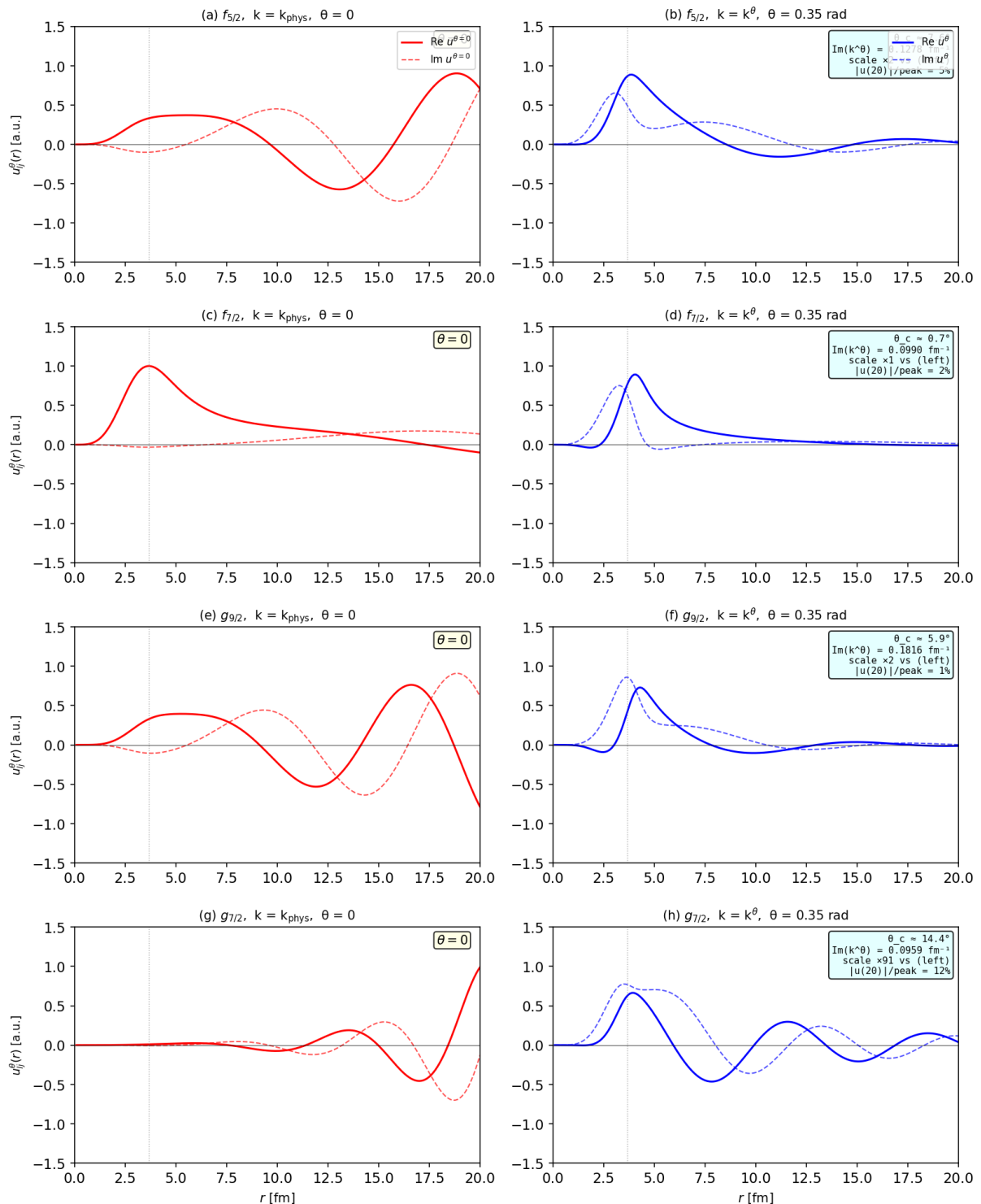


Fig. 7. Regular wave functions $u_{ij}^\theta(r; k_{res})$ at the resonance momentum for all four partial waves. Red: $\theta = 0$ (diverging, k_{phys}); blue: $\theta = 0.35$ rad (converging, k^θ). Solid: real part; dashed: imaginary part. Grey dotted vertical: nuclear radius $R \approx 3.66$ fm. All wave functions normalised to the peak of the $\theta = 0.35$ rad solution

The most striking feature is the qualitative difference in the large- r behaviour: when $\theta = 0$, the resonance wave function is evaluated with $k_{phys} = k^\theta e^{-i\theta}$ ($Im < 0$), which lies on the unphysical sheet and therefore grows exponentially at large r —it is not square-integrable. When $\theta = 0.35$ rad, the same physical resonance is captured with k^θ ($Im > 0$), placing the pole in the upper half k -plane; the wave function then decays at large r and is square-integrable.

For the narrow $f_{7/2}$ resonance ($\Gamma \approx 0.09$ MeV), $Im(k_{phys})$ is very small, so the divergence when $\theta = 0$ is slow. Conversely, for the broad $g_{9/2}$ resonance ($\Gamma \approx 4.7$ MeV), the $\theta = 0$ wave function rises more rapidly. When $\theta = 0.35$ rad, all four wave functions converge smoothly inside the nuclear radius ($r \lesssim R \approx 3.66$ fm) before decaying outward, directly illustrating why the CSM enables a square-integrable description of resonance states.

4 Conclusion

In this work, we applied the Complex Scaling Method within the Jost-function framework to study single-particle bound and resonance states in ^{24}O . The main results are as follows:

- Six bound states ($1s_{1/2}$, $1p_{3/2}$, $1p_{1/2}$, $1d_{5/2}$, $2s_{1/2}$, $1d_{3/2}$) with the shallowest at -3.64 MeV were determined.
- Four resonances above threshold, obtained via $E_{res} = \frac{\hbar^2}{2m}(k^\theta e^{-i\theta})^2$ from the ABC-stable θ steps: $f_{7/2}$ ($E_r \approx 1.85$ MeV, $\Gamma \approx 0.09$ MeV, all five steps), $f_{5/2}$ ($E_r \approx 7.02$ MeV, $\Gamma \approx 3.7$ MeV, $\theta \geq 0.14$ rad), $g_{9/2}$ ($E_r \approx 11.2$ MeV, $\Gamma \approx 4.8$ MeV, $\theta \geq 0.14$ rad), and $g_{7/2}$ ($E_r \approx 17.5$ MeV, $\Gamma \approx 19$ MeV, $\theta = 0.28$ and 0.35 rad) were found.
- The θ -independence of E_r across the ABC-stable range confirms the physical nature of the resonances (ABC theorem). For $g_{7/2}$, two θ steps (0.28 and 0.35 rad) exceed $\theta_c \approx 0.23$ rad, providing initial evidence for this state.

- The CSM avoids analytic continuation by rotating resonance poles into the upper half k -plane, making them directly accessible on the physical Riemann sheet. This is especially advantageous for very narrow resonances, such as $f_{7/2}$, which are difficult to resolve by conventional energy-grid scanning.

- Resonance energies identified through the T-matrix peak, level density maximum, and Newton-Raphson pole position are mutually consistent for all four channels, validating the CSM implementation.

- The regular wave functions at resonance directly illustrate the CSM mechanism: when $\theta = 0$, the wave function diverges at large r , while $\theta = 0.35$ rad, it decays smoothly and is square-integrable. For $g_{7/2}$, $\theta = 0.35$ rad gives a decay length ~ 10 fm, demonstrating clear convergence for all four channels.

- The spectra above are those of the Woods-Saxon potential of Table 1. Because that potential binds $1d_{3/2}$ while ^{25}O is unbound, they demonstrate the method rather than predict the ^{25}O spectrum.

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